Brazil National Olympiad 2016

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– Day 1

1 Let ABC be a triangle. r and s are the angle bisectors of $\angle ABC$ and $\angle BCA$, respectively.
The points E in r and D in s are such that $AD\|BE$ and $AE\|CD$.
The lines BD and CE cut each other at F. I is the incenter of ABC.

Show that if A,F,I are collinear, then $AB=AC$.

2 Find the smallest number n such that any set of n ponts in a Cartesian plan, all of them with integer coordinates, contains two poitns such that the square of its mutual distance is a multiple of 2016.

3 Let it k be a fixed positive integer. Alberto and Beralto play the following game:
Given an initial number N_0 and starting with Alberto, they alternately do the following operation: change the number n for a number m such that $m<n$ and m and n differ, in its base-2 representation, in exactly l consecutive digits for some l such that $1\leq l\leq k$.
If someone can't play, he loses.

We say a non-negative integer t is a *winner* number when the gamer who receives the number t has a winning strategy, that is, he can choose the next numbers in order to guarrantee his own victory, regardless the options of the other player.
Else, we call it *loser*.

Prove that, for every positive integer N, the total of non-negative loser integers lesser than 2^N is $2^{N-\lfloor \frac{log(min\{N,k\})}{log2} \rfloor}$

– Day 2

4 What is the greatest number of positive integers lesser than or equal to 2016 we can choose such that it doesn't have two of them differing by 1,2, or 6?

5 Consider the second-degree polynomial $P(x)=4x^2+12x-3015$. Define the sequence of polynomials $P_1(x)=\frac{P(x)}{2016}$ and $P_{n+1}(x)=\frac{P(P_n(x))}{2016}$ for every integer $n\geq 1$.

-Show that exists a real number r such that $P_n(r)<0$ for every positive integer n.
-Find how many integers m are such that $P_n(m)<0$ for infinite positive integers n.

6 Lei it $ABCD$ be a non-cyclical, convex quadrilateral, with no parallel sides.
The lines AB and CD meet in E.
Let it $M \neq E$ be the intersection of circumcircles of ADE and BCE.
The internal angle bisectors of $ABCD$ form an convex, cyclical quadrilateral with circumcenter I.
The external angle bisectors of $ABCD$ form an convex, cyclical quadrilateral with circumcenter J.

Show that I, J, M are colinear.
