Mexico National Olympiad 2016

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–	Day 1
1	Let C_1 and C_2 be two circumferences externally tangents at S such that the radius of C_2 is the triple of the radius of C_1. Let a line be tangent to C_1 at $P \neq S$ and to C_2 at $Q \neq S$. Let T be a point on C_2 such that QT is diameter of C_2. Let the angle bisector of $\angle SQT$ meet ST at R. Prove that $QR = RT$
2	A pair of positive integers m, n is called *guerrera*, if there exists positive integers a, b, c, d such that $m = ab$, $n = cd$ and $a + b = c + d$. For example the pair $8, 9$ is *guerrera* cause $8 = 4 \cdot 2$, $9 = 3 \cdot 3$ and $4 + 2 = 3 + 3$. We paint the positive integers if the following order: We start painting the numbers 3 and 5. If a positive integer x is not painted and a positive y is painted such that the pair x, y is *guerrera*, we paint x. Find all positive integers x that can be painted.
3	Find the minimum real x that satisfies $$\lfloor x \rfloor < \lfloor x^2 \rfloor < \lfloor x^3 \rfloor < \cdots < \lfloor x^n \rfloor < \lfloor x^{n+1} \rfloor < \cdots$$
–	Day 2
4	We say a non-negative integer n "*contains*" another non-negative integer m, if the digits of its decimal expansion appear consecutively in the decimal expansion of n. For example, 2016 *contains* 2, 0, 1, 6, 20, 16, 201, and 2016. Find the largest integer n that does not *contain* a multiple of 7.
5	The numbers from 1 to n^2 are written in order in a grid of $n \times n$, one number in each square, in such a way that the first row contains the numbers from 1 to n from left to right; the second row contains the numbers $n + 1$ to $2n$ from left to right, and so on and so forth. An allowed move on the grid consists in choosing any two adjacent squares (i.e. two squares that share a side), and add (or subtract) the same integer to both of the numbers that appear on those squares. Find all values of n for which it is possible to make every squares to display 0 after making any number of moves as necessary and, for those cases in which it is possible, find the minimum number of moves that are necessary to do this.

6 Let $ABCD$ a quadrilateral inscribed in a circumference, l_1 the parallel to BC through A, and l_2 the parallel to AD through B. The line DC intersects l_1 and l_2 at E and F, respectively. The perpendicular to l_1 through A intersects BC at P, and the perpendicular to l_2 through B cuts AD at Q. Let Γ_1 and Γ_2 be the circumferences that pass through the vertex of triangles ADE and BFC, respectively. Prove that Γ_1 and Γ_2 are tangent to each other if and only if DP is perpendicular to CQ.