National Math Olympiad (Second Round) 1989

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by Amir Hossein, Zeus93

Day 1

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1 **(a)** Let n be a positive integer, prove that

$$\sqrt{n+1}-\sqrt{n}<\frac{1}{2\sqrt{n}}$$

(b) Find a positive integer n for which

$$\left\lfloor 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{n}}\right\rfloor=12$$

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2 A sphere S with center O and radius R is given. Let P be a fixed point on this sphere. Points A, B, C move on the sphere S such that we have $\angle APB=\angle BPC=\angle CPA=90^\circ$. Prove that the plane of triangle ABC passes through a fixed point.

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3 Let $\{a_n\}_{n\geq 1}$ be a sequence in which $a_1=1$ and $a_2=2$ and

$$a_{n+1}=1+a_1a_2a_3\cdots a_{n-1}+(a_1a_2a_3\cdots a_{n-1})^2 \qquad \forall n\geq 2.$$

Prove that

$$\lim_{n\to\infty}\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\right)=2$$

Day 2

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1 In a sport competition, m teams have participated. We know that each two teams have competed exactly one time and the result is winning a team and losing the other team (i.e. there is no equal result). Prove that there exists a team x such that for each team y, either x wins y or there exists a team z for which x wins z and z wins y.

[i.e. prove that in every tournament there exists a king.]

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2 Let n be a positive integer. Prove that the polynomial

$$P(x)=\frac{x^n}{n!}+\frac{x^{n-1}}{(n-1)!}+\ldots+x+1$$

Does not have any rational root.

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3 A line d is called *faithful* to triangle ABC if d be in plane of triangle ABC and the reflections of d over the sides of ABC be concurrent. Prove that for any two triangles with acute angles lying in the same plane, either there exists exactly one *faithful* line to both of them, or there exist infinitely *faithful* lines to them.
