National Math Olympiad (Second Round) 1991

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by Amir Hossein

Day 1

1 Prove that the equation $x + x^2 = y + y^2 + y^3$ do not have any solutions in positive integers.

2 Let $ABCD$ be a tetragonal.

(a) If the plane (P) cuts $ABCD$, find the necessary and sufficient condition such that the area formed from the intersection of the plane (P) and the tetragonal be a parallelogram. Prove that the problem has three solutions in this case.

(b) Consider one of the solutions of **(a)**. Find the situation of the plane (P) for which the parallelogram has maximum area.

(c) Find a plane (P) for which the parallelogram be a lozenge and then find the length side of his lozenge in terms of the length of the edges of $ABCD$.

3 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 1$ and

$$f(x + y) = f(x) + f(y)$$

And for all $x \in \mathbb{R}/\{0\}$ we have $f\left(\frac{1}{x}\right) = \frac{1}{f(x)}$. Find all such functions f.

Day 2

1 Prove that there exist at least six points with rational coordinates on the curve of the equation

$$y^3 = x^3 + x + 1370^{1370}$$

2 Triangle ABC is inscribed in circle C. The bisectors of the angles A, B and C meet the circle C again at the points A', B', C'. Let I be the incenter of ABC, prove that

$$\frac{IA'}{IA} + \frac{IB'}{IB} + \frac{IC'}{IC} \geq 3$$

$$, IA' + IB' + IC' \geq IA + IB + IC$$

3 Three groups A, B and C of mathematicians from different countries have invited to a ceremony. We have formed meetings such that three mathematicians participate in every meeting and there is exactly one mathematician from each group in every meeting. Also every two mathematicians have participated in exactly one meeting with each other.

(a) Prove that if this is possible, then number of mathematicians of the groups is equal.

(b) Prove that if there exist 3 mathematicians in each group, then that work is possible.

(c) Prove that if number mathematicians of the groups be equal, then that work is possible.