National Math Olympiad (Second Round) 1993

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by Peter, Vaf, Amir Hossein

Day 1

1	Suppose that p is a prime number and is greater than 3. Prove that $7^p - 6^p - 1$ is divisible by 43.
2	Let ABC be an acute triangle with sides and area equal to a, b, c and S respectively. Prove or disprove that a necessary and sufficient condition for existence of a point P inside the triangle ABC such that the distance between P and the vertices of ABC be equal to x, y and z respectively is that there be a triangle with sides a, y, z and area S_1, a triangle with sides b, z, x and area S_2 and a triangle with sides c, x, y and area S_3 where $S_1 + S_2 + S_3 = S$.
3	Let n, r be positive integers. Find the smallest positive integer m satisfying the following condition. For each partition of the set $\{1, 2, \dots, m\}$ into r subsets $A_1, A_2, \dots, A_r$, there exist two numbers a and b in some $A_i, 1 \leq i \leq r$, such that $$1 < \frac{a}{b} < 1 + \frac{1}{n}.$$

Day 2

1	G is a graph with n vertices $A_1, A_2, \dots, A_n$, such that for each pair of non adjacent vertices A_i and A_j , there exist another vertex A_k that is adjacent to both A_i and A_j. **(a)** Find the minimum number of edges in such a graph. **(b)** If $n = 6$ and A_1, A_2, A_3, A_4, A_5, and A_6 form a cycle of length 6, find the number of edges that must be added to this cycle such that the above condition holds.
2	Show that if D_1 and D_2 are two skew lines, then there are infinitely many straight lines such that their points have equal distance from D_1 and D_2.
3	Let $f(x)$ and $g(x)$ be two polynomials with real coefficients such that for infinitely many rational values of x, the fraction $\frac{f(x)}{g(x)}$ is rational. Prove that $\frac{f(x)}{g(x)}$ can be written as the ratio of two polynomials with rational coefficients.