National Math Olympiad (Second Round) 1995

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Day 1

1 Prove that for every positive integer $n \geq 3$ there exist two sets $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_n\}$ for which

i) $A \cap B = \varnothing$.

ii) $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$.

ii) $x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2$.

2 Let ABC be an acute triangle and let ℓ be a line in the plane of triangle ABC. We've drawn the reflection of the line ℓ over the sides AB, BC and AC and they intersect in the points A', B' and C'. Prove that the incenter of the triangle $A'B'C'$ lies on the circumcircle of the triangle ABC.

3 Let k be a positive integer. $12k$ persons have participated in a party and everyone shake hands with $3k + 6$ other persons. We know that the number of persons who shake hands with every two persons is a fixed number. Find k.

Day 2

1 Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.
(For example, $23 = 9 + 8 + 6.$)

2 Let $n \geq 0$ be an integer. Prove that

$$\lceil \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} \rceil = \lceil \sqrt{9n+8} \rceil$$

Where $\lceil x \rceil$ is the smallest integer which is greater or equal to x.

3 In a quadrilateral $ABCD$ let A', B', C' and D' be the circumcenters of the triangles BCD, CDA, DAB and ABC, respectively. Denote by $S(X, YZ)$ the plane which passes through the point X and is perpendicular to the line YZ. Prove that if A', B', C' and D' don't lie in a plane, then four planes $S(A, C'D'), S(B, A'D'), S(C, A'B')$ and $S(D, B'C')$ pass through a common point.