European Mathematical Cup 2015

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by Eray, Ankoganit

– Junior Division

1 We are given an $n \times n$ board. Rows are labeled with numbers 1 to n downwards and columns are labeled with numbers 1 to n from left to right. On each field we write the number $x^2 + y^2$ where (x, y) are its coordinates. We are given a figure and can initially place it on any field. In every step we can move the figure from one field to another if the other field has not already been visited and if at least one of the following
conditions is satisfied:
- the numbers in those 2 fields give the same remainders when divided by n,
- those fields are point reflected with respect to the center of the board.Can all the fields be visited in case:
- $n = 4$,
- $n = 5$?

Josip Pupi

2 Let m, n, p be fixed positive real numbers which satisfy $mnp = 8$. Depending on these constants, find the minimum of

$$x^2 + y^2 + z^2 + mxy + nxz + pyz,$$

where x, y, z are arbitrary positive real numbers satisfying $xyz = 8$. When is the equality attained?
Solve the problem for:
-$m = n = p = 2,$
- arbitrary (but fixed) positive real numbers m, n, p.

Stijn Cambie

3 Let $d(n)$ denote the number of positive divisors of n. For positive integer n we define $f(n)$ as

$$f(n) = d(k_1) + d(k_2) + \cdots + d(k_m),$$

where $1 = k_1 < k_2 < \cdots < k_m = n$ are all divisors of the number n. We call an integer $n > 1$ *almost perfect* if $f(n) = n$. Find all almost perfect numbers.

Paulius Avydis

4 Let ABC be an acute angled triangle. Let B', A' be points on the perpendicular bisectors of AC, BC respectively such that $B'A \perp AB$ and $A'B \perp AB$. Let P be a point on the segment

AB and O the circumcenter of the triangle ABC. Let D, E be points on BC, AC respectively such that $DP \perp BO$ and $EP \perp AO$. Let O' be the circumcenter of the triangle CDE. Prove that B', A' and O' are collinear.

Steve Dinh

– Senior Division

1 $A = \{a, b, c\}$ is a set containing three positive integers. Prove that we can find a set $B \subset A$, $B = \{x, y\}$ such that for all odd positive integers m, n we have

$$10 \mid x^m y^n - x^n y^m.$$

Tomi Dimovski

2 Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a+b+c+3}{4} \geqslant \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}.$$

Dimitar Trenevski

3 Circles k_1 and k_2 intersect in points A and B, such that k_1 passes through the center O of the circle k_2. The line p intersects k_1 in points K and O and k_2 in points L and M, such that the point L is between K and O. The point P is orthogonal projection of the point L to the line AB. Prove that the line KP is parallel to the $M-$median of the triangle ABM.

Matko Ljulj

4 A group of mathematicians is attending a conference. We say that a mathematician is $k-$*content* if he is in a room with at least k people he admires or if he is admired by at least k other people in the room. It is known that when all participants are in a same room then they are all at least $3k+1$-content. Prove that you can assign everyone into one of 2 rooms in a way that everyone is at least k-content in his room and neither room is empty. *Admiration is not necessarily mutual and no one admires himself.*

Matija Buci