**National Math Olympiad (Second Round) 2000**

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**Day 1**

**1** $21$ distinct numbers are chosen from the set $\{1, 2, 3, \dots, 2046\}$. Prove that we can choose three distinct numbers $a, b, c$ among those $21$ numbers such that

$$bc < 2a^2 < 4bc$$

**2** The points $D, E$ and $F$ are chosen on the sides $BC$, $AC$ and $AB$ of triangle $ABC$, respectively. Prove that triangles $ABC$ and $DEF$ have the same centroid if and only if

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}$$

**3** Let $M = \{1, 2, 3, \dots, 10000\}$. Prove that there are $16$ subsets of $M$ such that for every $a \in M$, there exist $8$ of those subsets that intersection of the sets is exactly $\{a\}$.

**Day 2**

**1** Find all positive integers $n$ such that we can divide the set $\{1, 2, 3, \dots, n\}$ into three sets with the same sum of members.

**2** In a tetrahedron we know that sum of angles of all vertices is $180^\circ$. (e.g. for vertex $A$, we have $\angle BAC + \angle CAD + \angle DAB = 180^\circ$.)
Prove that faces of this tetrahedron are four congruent triangles.

**3** *Super number* is a sequence of numbers $0, 1, 2, \dots, 9$ such that it has infinitely many digits at left. For example $\dots 3030304$ is a *super number*. Note that all of positive integers are *super numbers*, which have zeros before they're original digits (for example we can represent the number $4$ as $\dots, 00004$). Like positive integers, we can add up and multiply *super numbers*. For example:

$$\begin{array}{r} \dots 3030304 \\ +\dots 4571378 \\ \hline \dots 7601682 \end{array}$$

And

$$\begin{array}{r} \dots 3030304 \\ \times \dots 4571378 \\ \hline \dots 4242432 \\ \dots 212128 \\ \dots 90912 \\ \dots 0304 \\ \dots 128 \\ \dots 20 \\ \dots 6 \\ \hline \dots 5038912 \end{array}$$

**a)** Suppose that $A$ is a *super number*. Prove that there exists a *super number* $B$ such that $A + B = \overleftarrow{0}$ (Note: $\overleftarrow{0}$ means a super number that all of its digits are zero).

**b)** Find all *super numbers* $A$ for which there exists a *super number* $B$ such that $A \times B = \overleftarrow{0}\,1$ (Note: $\overleftarrow{0}\,1$ means the super number $\dots 00001$).

**c)** Is this true that if $A \times B = \overleftarrow{0}$, then $A = \overleftarrow{0}$ or $B = \overleftarrow{0}$? Justify your answer.