National Math Olympiad (Second Round) 2002

www.artofproblemsolving.com/community/c3886

by Bugi, Amir Hossein, lomos_lupin

1 Let $n \in \mathbb{N}$ and A_n set of all permutations $(a_1, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$ for which

$$k|2(a_1 + \cdots + a_k), \text{ for all } 1 \leq k \leq n.$$

Find the number of elements of the set A_n.

Proposed by Vidan Govedarica, Serbia

---

2 A rectangle is partitioned into finitely many small rectangles. We call a point a cross point if it belongs to four different small rectangles. We call a segment on the obtained diagram maximal if there is no other segment containing it. Show that the number of maximal segments plus the number of cross points is 3 more than the number of small rectangles.

---

3 In a convex quadrilateral $ABCD$ with $\angle ABC = \angle ADC = 135^\circ$, points M and N are taken on the rays AB and AD respectively such that $\angle MCD = \angle NCB = 90^\circ$. The circumcircles of triangles AMN and ABD intersect at A and K. Prove that $AK \perp KC$.

---

4 Let A and B be two fixed points in the plane. Consider all possible convex quadrilaterals $ABCD$ with $AB = BC, AD = DC$, and $\angle ADC = 90^\circ$. Prove that there is a fixed point P such that, for every such quadrilateral $ABCD$ on the same side of AB, the line DC passes through P.

---

5 Let δ be a symbol such that $\delta \neq 0$ and $\delta^2 = 0$. Define $\mathbb{R}[\delta] = \{a + b\delta | a, b \in \mathbb{R}\}$, where $a + b\delta = c + d\delta$ if and only if $a = c$ and $b = d$, and define

$$(a + b\delta) + (c + d\delta) = (a + c) + (b + d)\delta,$$

$$(a + b\delta) \cdot (c + d\delta) = ac + (ad + bc)\delta.$$

Let $P(x)$ be a polynomial with real coefficients. Show that $P(x)$ has a multiple real root if and only if $P(x)$ has a non-real root in $\mathbb{R}[\delta]$.

---

6 Let G be a simple graph with 100 edges on 20 vertices. Suppose that we can choose a pair of disjoint edges in 4050 ways. Prove that G is regular.
