National Math Olympiad (Second Round) 2008

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Day 1

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1 In how many ways, can we draw $n-3$ diagonals of a n-gon with equal sides and equal angles such that: $i)$ none of them intersect each other in the polygonal. $ii)$ each of the produced triangles has at least one common side with the polygonal.

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2 Let I_a be the A-excenter of ΔABC and the A-excircle of ΔABC be tangent to the lines AB, AC at B', C', respectively. I_aB, I_aC meet $B'C'$ at P, Q, respectively. M is the meet point of BQ, CP. Prove that the length of the perpendicular from M to BC is equal to r where r is the radius of incircle of ΔABC.

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3 Let $a, b, c,$ and d be real numbers such that at least one of c and d is non-zero. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as $f(x) = \frac{ax+b}{cx+d}$. Suppose that for all $x \in \mathbb{R}$, we have $f(x) \neq x$. Prove that if there exists some real number a for which $f^{1387}(a) = a$, then for all x in the domain of f^{1387}, we have $f^{1387}(x) = x$. Notice that in this problem,

$$f^{1387}(x) = \underbrace{f(f(\cdots(f(x)))\cdots)}_{\text{1387 times}}.$$

Hint. Prove that for every function $g(x) = \frac{sx+t}{ux+v}$, if the equation $g(x) = x$ has more than 2 roots, then $g(x) = x$ for all $x \in \mathbb{R} - \left\{\frac{-v}{u}\right\}$.

Day 2

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1 $\mathbb{N}$ is the set of positive integers and $a \in \mathbb{N}$. We know that for every $n \in \mathbb{N}$, $4(a^n + 1)$ is a perfect cube. Prove that $a = 1$.

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2 We want to choose telephone numbers for a city. The numbers have 10 digits and 0 isnt used in the numbers. Our aim is: We dont choose some numbers such that every 2 telephone numbers are different in more than one digit OR every 2 telephone numbers are different in a digit which is more than 1. What is the maximum number of telephone numbers which can be chosen? In how many ways, can we choose the numbers in this maximum situation?

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3 In triangle ABC, H is the foot of perpendicular from A to BC. O is the circumcenter of ΔABC. T, T' are the feet of perpendiculars from H to AB, AC, respectively. We know that $AC = 2OT$. Prove that $AB = 2OT'$.
