National Math Olympiad (Second Round) 2009

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by sororak

Day 1

1 Let $p(x)$ be a quadratic polynomial for which :

$$|p(x)| \leq 1 \qquad \forall x \in \{-1, 0, 1\}$$

Prove that:

$$|p(x)| \leq \frac{5}{4} \qquad \forall x \in [-1, 1]$$

2 In some of the 1×1 squares of a square garden 50×50 we've grown apple, pomegranate and peach trees (At most one tree in each square). We call a 1×1 square a *room* and call two rooms *neighbor* if they have one common side. We know that a pomegranate tree has at least one apple neighbor room and a peach tree has at least one apple neighbor room and one pomegranate neighbor room. We also know that an empty room (a room in which theres no trees) has at least one apple neighbor room and one pomegranate neighbor room and one peach neighbor room.
Prove that the number of empty rooms is not greater than 1000.

3 Let ABC be a triangle and the point D is on the segment BC such that AD is the interior bisector of $\angle A$. We stretch AD such that it meets the circumcircle of ΔABC at M. We draw a line from D such that it meets the lines MB, MC at P, Q, respectively (M is not between B, P and also is not between C, Q).
Prove that $\angle PAQ \geq \angle BAC$.

Day 2

1 We have a $(n+2) \times n$ rectangle and weve divided it into $n(n+2)$ 1×1 squares. $n(n+2)$ soldiers are standing on the intersection points ($n+2$ rows and n columns). The commander shouts and each soldier stands on its own location or gaits one step to north, west, east or south so that he stands on an adjacent intersection point. After the shout, we see that the soldiers are standing on the intersection points of a $n \times (n+2)$ rectangle (n rows and $n+2$ columns) such that the first and last row are deleted and 2 columns are added to the right and left (To the left 1 and 1 to the right).
Prove that n is even.

2 Let $a_1 < a_2 < \cdots < a_n$ be positive integers such that for every distinct $1 \le i, j \le n$ we have $a_j - a_i$ divides a_i.
Prove that

$$ia_j \le ja_i \qquad \text{for } 1 \le i < j \le n$$

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3 11 people are sitting around a circle table, orderly (means that the distance between two adjacent persons is equal to others) and 11 cards with numbers 1 to 11 are given to them. Some may have no card and some may have more than 1 card. In each round, one [and only one] can give one of his cards with number i to his adjacent person if after and before the round, the locations of the cards with numbers $i-1, i, i+1$ dont make an acute-angled triangle.
(Card with number 0 means the card with number 11 and card with number 12 means the card with number 1!)
Suppose that the cards are given to the persons regularly clockwise. (Mean that the number of the cards in the clockwise direction is increasing.)
Prove that the cards cant be gathered at one person.
