National Math Olympiad (Second Round) 2012

www.artofproblemsolving.com/community/c3896

by goldeneagle, goodar2006

Day 1

---

1 Consider a circle C_1 and a point O on it. Circle C_2 with center O, intersects C_1 in two points P and Q. C_3 is a circle which is externally tangent to C_2 at R and internally tangent to C_1 at S and suppose that RS passes through Q. Suppose X and Y are second intersection points of PR and OR with C_1. Prove that QX is parallel with SY.

---

2 Suppose n is a natural number. In how many ways can we place numbers $1, 2, ..., n$ around a circle such that each number is a divisor of the sum of it's two adjacent numbers?

---

3 Prove that if t is a natural number then there exists a natural number $n > 1$ such that $(n, t) = 1$ and none of the numbers $n + t, n^2 + t, n^3 + t, ...$ are perfect powers.

Day 2

---

1 **a)** Do there exist 2-element subsets $A_1, A_2, A_3, ...$ of natural numbers such that each natural number appears in exactly one of these sets and also for each natural number n, sum of the elements of A_n equals $1391 + n$?

b) Do there exist 2-element subsets $A_1, A_2, A_3, ...$ of natural numbers such that each natural number appears in exactly one of these sets and also for each natural number n, sum of the elements of A_n equals $1391 + n^2$?

Proposed by Morteza Saghafian

---

2 Consider the second degree polynomial $x^2 + ax + b$ with real coefficients. We know that the necessary and sufficient condition for this polynomial to have roots in real numbers is that its discriminant, $a^2 - 4b$ be greater than or equal to zero. Note that the discriminant is also a polynomial with variables a and b. Prove that the same story is not true for polynomials of degree 4: Prove that there does not exist a 4 variable polynomial $P(a, b, c, d)$ such that:

The fourth degree polynomial $x^4 + ax^3 + bx^2 + cx + d$ can be written as the product of four 1st degree polynomials if and only if $P(a, b, c, d) \geq 0$. (All the coefficients are real numbers.)

Proposed by Sahand Seifnashri

---

3 The incircle of triangle ABC, is tangent to sides BC, CA and AB in D, E and F respectively. The reflection of F with respect to B and the reflection of E with respect to C are T and S

respectively. Prove that the incenter of triangle AST is inside or on the incircle of triangle ABC.

Proposed by Mehdi E'tesami Fard
