National Math Olympiad (Second Round) 2013

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Day 1

1 Find all pairs (a, b) of positive integers for which $\gcd(a, b) = 1$, and $\frac{a}{b} = \overline{b.a}$. (For example, if $a = 92$ and $b = 13$, then $b/a = 13.92$)

2 Let n be a natural number and suppose that $w_1, w_2, \ldots, w_n$ are n weights . We call the set of $\{w_1, w_2, \ldots, w_n\}$ to be a *Perfect Set* if we can achieve all of the $1, 2, \ldots, W$ weights with sums of $w_1, w_2, \ldots, w_n$, where $W = \sum_{i=1}^{n} w_i$. Prove that if we delete the maximum weight of a Perfect Set, the other weights make again a Perfect Set.

3 Let M be the midpoint of (the smaller) arc BC in circumcircle of triangle ABC. Suppose that the altitude drawn from A intersects the circle at N. Draw two lines through circumcenter O of ABC paralell to MB and MC, which intersect AB and AC at K and L, respectively. Prove that $NK = NL$.

Day 2

1 Let P be a point out of circle C. Let PA and PB be the tangents to the circle drawn from C. Choose a point K on AB . Suppose that the circumcircle of triangle PBK intersects C again at T. Let P' be the reflection of P with respect to A. Prove that

$$\angle PBT = \angle P'KA$$

2 Suppose a $m \times n$ table. We write an integer in each cell of the table. In each move, we chose a column, a row, or a diagonal (diagonal is the set of cells which the difference between their row number and their column number is constant) and add either $+1$ or -1 to all of its cells. Prove that if for all arbitrary 3×3 table we can change all numbers to zero, then we can change all numbers of $m \times n$ table to zero.

(*Hint*: First of all think about it how we can change number of 3×3 table to zero.)

3 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers for which

$$a_{n+2} = \left[\frac{2a_n}{a_{n+1}}\right] + \left[\frac{2a_{n+1}}{a_n}\right].$$

Prove that there exists a positive integer m such that $a_m = 4$ and $a_{m+1} \in \{3, 4\}$.

Note. $[x]$ is the greatest integer not exceeding x.