National Math Olympiad (Second Round) 2014

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Day 1 April 1st

1 A basket is called *"Stuff Basket"* if it includes 10 kilograms of rice and 30 number of eggs. A market is to distribute 100 Stuff Baskets. We know that there is totally 1000 kilograms of rice and 3000 number of eggs in the baskets, but some of market's baskets include either more or less amount of rice or eggs. In each step, market workers can select two baskets and move an arbitrary amount of rice or eggs between selected baskets. Starting from an arbitrary situation, what's the minimum number of steps that workers provide 100 Stuff Baskets?

2 Let $ABCD$ be a square. Let N, P be two points on sides AB, AD, respectively such that $NP = NC$, and let Q be a point on AN such that $\angle QPN = \angle NCB$. Prove that

$$\angle BCQ = \frac{1}{2}\angle AQP.$$

3 Let x, y, z be three non-negative real numbers such that

$$x^2 + y^2 + z^2 = 2(xy + yz + zx).$$

Prove that

$$\frac{x + y + z}{3} \geq \sqrt[3]{2xyz}.$$

Day 2 April 2nd

1 Find all positive integers (m, n) such that

$$n^{n^n} = m^m.$$

2 A subset S of positive real numbers is called *powerful* if for any two distinct elements a, b of S, at least one of a^b or b^a is also an element of S.
a) Give an example of a four elements powerful set.
b) Prove that every finite powerful set has at most four elements.

3 Members of "Professionous Riddlous" society have been divided into some groups, and groups are changed in a special way each weekend: In each group, one of the members is specified as the best member, and the best members of all groups separate from their previous group and form a new group. If a group has only one member, that member joins the new group and the previous group will be removed. Suppose that the society has n members at first, and all the members are in one group. Prove that a week will come, after which number of members of each group will be at most $1 + \sqrt{2n}$.