BMO TST 2009

www.artofproblemsolving.com/community/c3906

by ridgers

1 Given the equation $x^4 - x^3 - 1 = 0$
(a) Find the number of its real roots.
(b) We denote by S the sum of the real roots and by P their product. Prove that $P < -\frac{11}{10}$ and $S > \frac{6}{11}$.

---

2 Let C_1 and C_2 be concentric circles, with C_2 in the interior of C_1. From a point A on C_1, draw the tangent AB to C_2 $(B \in C_2)$. Let C be the second point of intersection of AB and C_1,and let D be the midpoint of AB. A line passing through A intersects C_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

This question is taken from Mathematical Olympiad Challenges , the 9-th exercise in 1.3 Power of a Point.

---

3 For the give functions in $\mathbb{N}$:
(a) Euler's ϕ function ($\phi(n)$- the number of natural numbers smaller than n and coprime with n);
(b) the σ function such that the $\sigma(n)$ is the sum of natural divisors of n.
solve the equation $\phi(\sigma(2^x)) = 2^x$.

---

4 Find all the polynomials $P(x)$ of a degree $\leq n$ with real non-negative coefficients such that $P(x) \cdot P(\frac{1}{x}) \leq [P(1)]^2$, $\forall x > 0$.
