BMO TST 2014

www.artofproblemsolving.com/community/c3909

by Olemissmath

1 Prove that for $n \geq 2$ the following inequality holds:

$$\frac{1}{n+1}\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right).$$

---

2 Solve the following equation in $\mathbb{R}$:

$$\left(x-\frac{1}{x}\right)^{\frac{1}{2}}+\left(1-\frac{1}{x}\right)^{\frac{1}{2}}=x.$$

---

3 From the point P outside a circle ω with center O draw the tangents PA and PB where A and B belong to ω.In a random point M in the chord AB we draw the perpendicular to OM, which intersects PA and PB in C and D. Prove that M is the midpoint CD.

---

4 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)f(y)=f(x+y)+xy$ for all $x,y\in\mathbb{R}$.

---

5 Find all non-negative integers k,n which satisfy $2^{2k+1}+9\cdot 2^k+5=n^2$.
