**IMO Shortlist 1972**

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**1** $f$ and $g$ are real-valued functions defined on the real line. For all $x$ and $y$, $f(x+y)+f(x-y) = 2f(x)g(y)$. $f$ is not identically zero and $|f(x)| \leq 1$ for all $x$. Prove that $|g(x)| \leq 1$ for all $x$.

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**2** We are given $3n$ points $A_1, A_2, \ldots, A_{3n}$ in the plane, no three of them collinear. Prove that one can construct $n$ disjoint triangles with vertices at the points $A_i$.

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**3** The least number is $m$ and the greatest number is $M$ among $a_1, a_2, \ldots, a_n$ satisfying $a_1 + a_2 + ... + a_n = 0$. Prove that

$$a_1^2 + \cdots + a_n^2 \leq -nmM$$

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**4** Let $n_1, n_2$ be positive integers. Consider in a plane $E$ two disjoint sets of points $M_1$ and $M_2$ consisting of $2n_1$ and $2n_2$ points, respectively, and such that no three points of the union $M_1 \cup M_2$ are collinear. Prove that there exists a straightline $g$ with the following property: Each of the two half-planes determined by $g$ on $E$ ($g$ not being included in either) contains exactly half of the points of $M_1$ and exactly half of the points of $M_2$.

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**5** Prove the following assertion: The four altitudes of a tetrahedron $ABCD$ intersect in a point if and only if

$$AB^2 + CD^2 = BC^2 + AD^2 = CA^2 + BD^2.$$

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**6** Show that for any $n \not\equiv 0 \pmod{10}$ there exists a multiple of $n$ not containing the digit $0$ in its decimal expansion.

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**7** Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

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**8** Prove that $(2m)!(2n)!$ is a multiple of $m!n!(m+n)!$ for any non-negative integers $m$ and $n$.

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**9** Find all positive real solutions to:

$$\begin{array}{rcl}(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) & \leq & 0 \\ (x_2^2 - x_4x_1)(x_3^2 - x_4x_1) & \leq & 0 \\ (x_3^2 - x_5x_2)(x_4^2 - x_5x_2) & \leq & 0 \\ (x_4^2 - x_1x_3)(x_5^2 - x_1x_3) & \leq & 0 \\ (x_5^2 - x_2x_4)(x_1^2 - x_2x_4) & \leq & 0\end{array}$$

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**10** Given $n > 4$, prove that every cyclic quadrilateral can be dissected into $n$ cyclic quadrilaterals.

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**11** Consider a sequence of circles $K_1, K_2, K_3, K_4, \dots$ of radii $r_1, r_2, r_3, r_4, \dots$, respectively, situated inside a triangle $ABC$. The circle $K_1$ is tangent to $AB$ and $AC$; $K_2$ is tangent to $K_1$, $BA$, and $BC$; $K_3$ is tangent to $K_2$, $CA$, and $CB$; $K_4$ is tangent to $K_3$, $AB$, and $AC$; etc.
(a) Prove the relation

$$r_1 \cot \frac{1}{2}A + 2\sqrt{r_1r_2} + r_2 \cot \frac{1}{2}B = r\left(\cot \frac{1}{2}A + \cot \frac{1}{2}B\right)$$

where $r$ is the radius of the incircle of the triangle $ABC$. Deduce the existence of a $t_1$ such that

$$r_1 = r \cot \frac{1}{2}B \cot \frac{1}{2}C \sin^2 t_1$$

(b) Prove that the sequence of circles $K_1, K_2, \dots$ is periodic.

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**12** Prove that from a set of ten distinct two-digit numbers, it is always possible to find two disjoint subsets whose members have the same sum.

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