IMO Shortlist 1973

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by Amir Hossein, orl

1 Let a tetrahedron $ABCD$ be inscribed in a sphere S. Find the locus of points P inside the sphere S for which the equality

$$\frac{AP}{PA_1}+\frac{BP}{PB_1}+\frac{CP}{PC_1}+\frac{DP}{PD_1}=4$$

holds, where A_1, B_1, C_1, and D_1 are the intersection points of S with the lines AP, BP, CP, and DP, respectively.

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2 Given a circle K, find the locus of vertices A of parallelograms $ABCD$ with diagonals $AC \leq BD$, such that BD is inside K.

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3 Prove that the sum of an odd number of vectors of length 1, of common origin O and all situated in the same semi-plane determined by a straight line which goes through $O,$ is at least 1.

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4 Let P be a set of 7 different prime numbers and C a set of 28 different composite numbers each of which is a product of two (not necessarily different) numbers from P. The set C is divided into 7 disjoint four-element subsets such that each of the numbers in one set has a common prime divisor with at least two other numbers in that set. How many such partitions of C are there ?

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5 A circle of radius 1 is located in a right-angled trihedron and touches all its faces. Find the locus of centers of such circles.

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6 Establish if there exists a finite set M of points in space, not all situated in the same plane, so that for any straight line d which contains at least two points from M there exists another straight line d', parallel with $d,$ but distinct from $d,$ which also contains at least two points from M.

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7 Given a tetrahedron $ABCD$, let $x = AB \cdot CD$, $y = AC \cdot BD$, and $z = AD \cdot BC$. Prove that there exists a triangle with edges $x, y, z.$

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8 Prove that there are exactly $\binom{k}{[k/2]}$ arrays $a_1, a_2, \ldots, a_{k+1}$ of nonnegative integers such that $a_1 = 0$ and $|a_i-a_{i+1}| = 1$ for $i = 1, 2, \ldots, k.$

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9 Let Ox, Oy, Oz be three rays, and G a point inside the trihedron $Oxyz$. Consider all planes passing through G and cutting Ox, Oy, Oz at points A, B, C, respectively. How is the plane to be placed in order to yield a tetrahedron $OABC$ with minimal perimeter ?

10 Let $a_1, \dots, a_n$ be n positive numbers and $0 < q < 1$. Determine n positive numbers $b_1, \dots, b_n$ so that:

a.) $a_k < b_k$ for all $k = 1, \dots, n$,
b.) $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for all $k = 1, \dots, n-1$,
c.) $\sum\limits_{k=1}^{n} b_k < \frac{1+q}{1-q} \cdot \sum\limits_{k=1}^{n} a_k$.

11 Determine the minimum value of $a^2 + b^2$ when (a, b) traverses all the pairs of real numbers for which the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

has at least one real root.

12 Consider the two square matrices

$$A = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & -1 & +1 \\ +1 & +1 & -1 & +1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & -1 & -1 \\ +1 & +1 & -1 & +1 & -1 \\ +1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & +1 \end{bmatrix}$$

with entries $+1$ and -1. The following operations will be called elementary:

(1) Changing signs of all numbers in one row;

(2) Changing signs of all numbers in one column;

(3) Interchanging two rows (two rows exchange their positions);

(4) Interchanging two columns.

Prove that the matrix B cannot be obtained from the matrix A using these operations.

13 Find the sphere of maximal radius that can be placed inside every tetrahedron that has all altitudes of length greater than or equal to 1.

14 A soldier needs to check if there are any mines in the interior or on the sides of an equilateral triangle ABC. His detector can detect a mine at a maximum distance equal to half the height of the triangle. The soldier leaves from one of the vertices of the triangle. Which is the minimum distance that he needs to traverse so that at the end of it he is sure that he completed successfully his mission?

15 Prove that for all $n \in \mathbb{N}$ the following is true:

$$2^n \prod_{k=1}^{n} \sin \frac{k\pi}{2n+1} = \sqrt{2n+1}$$

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16 Given $a, \theta \in \mathbb{R}, m \in \mathbb{N}$, and $P(x) = x^{2m} - 2|a|^m x^m \cos\theta + a^{2m}$, factorize $P(x)$ as a product of m real quadratic polynomials.

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17 G is a set of non-constant functions f. Each f is defined on the real line and has the form $f(x) = ax + b$ for some real a, b. If f and g are in G, then so is fg, where fg is defined by $fg(x) = f(g(x))$. If f is in G, then so is the inverse f^{-1}. If $f(x) = ax + b$, then $f^{-1}(x) = \frac{x-b}{a}$. Every f in G has a fixed point (in other words we can find x_f such that $f(x_f) = x_f$. Prove that all the functions in G have a common fixed point.
