IMO Shortlist 1974

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1 Three players A, B and C play a game with three cards and on each of these 3 cards it is written a positive integer, all 3 numbers are different. A game consists of shuffling the cards, giving each player a card and each player is attributed a number of points equal to the number written on the card and then they give the cards back. After a number (≥ 2) of games we find out that A has 20 points, B has 10 points and C has 9 points. We also know that in the last game B had the card with the biggest number. Who had in the first game the card with the second value (this means the middle card concerning its value).

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2 Prove that the squares with sides $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$ may be put into the square with side $\frac{3}{2}$ in such a way that no two of them have any interior point in common.

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3 Let $P(x)$ be a polynomial with integer coefficients. We denote $\deg(P)$ its degree which is ≥ 1. Let $n(P)$ be the number of all the integers k for which we have $(P(k))^2 = 1$. Prove that $n(P) - \deg(P) \leq 2$.

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4 The sum of the squares of five real numbers a_1, a_2, a_3, a_4, a_5 equals 1. Prove that the least of the numbers $(a_i - a_j)^2$, where $i, j = 1, 2, 3, 4, 5$ and $i \neq j$, does not exceed $\frac{1}{10}$.

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5 Let A_r, B_r, C_r be points on the circumference of a given circle S. From the triangle $A_rB_rC_r$, called Δ_r, the triangle Δ_{r+1} is obtained by constructing the points $A_{r+1}, B_{r+1}, C_{r+1}$on S such that $A_{r+1}A_r$ is parallel to B_rC_r, $B_{r+1}B_r$ is parallel to C_rA_r, and $C_{r+1}C_r$ is parallel to A_rB_r. Each angle of Δ_1 is an integer number of degrees and those integers are not multiples of 45. Prove that at least two of the triangles $\Delta_1, \Delta_2, \ldots, \Delta_{15}$ are congruent.

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6 Prove that for any n natural, the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

cannot be divided by 5.

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7 Let a_i, b_i be coprime positive integers for $i = 1, 2, \ldots, k$, and m the least common multiple of $b_1, \ldots, b_k$. Prove that the greatest common divisor of $a_1 \frac{m}{b_1}, \ldots, a_k \frac{m}{b_k}$ equals the greatest common divisor of $a_1, \ldots, a_k$.

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8 The variables $a, b, c, d,$ traverse, independently from each other, the set of positive real values.

What are the values which the expression

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

takes?

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9 Let x, y, z be real numbers each of whose absolute value is different from $\frac{1}{\sqrt{3}}$ such that $x+y+z = xyz$. Prove that

$$\frac{3x-x^3}{1-3x^2} + \frac{3y-y^3}{1-3y^2} + \frac{3z-z^3}{1-3z^2} = \frac{3x-x^3}{1-3x^2} \cdot \frac{3y-y^3}{1-3y^2} \cdot \frac{3z-z^3}{1-3z^2}$$

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10 Let ABC be a triangle. Prove that there exists a point D on the side AB of the triangle ABC, such that CD is the geometric mean of AD and DB, iff the triangle ABC satisfies the inequality $\sin A \sin B \le \sin^2 \frac{C}{2}$.

Alternative formulation, from IMO ShortList 1974, Finland 2: We consider a triangle ABC. Prove that: $\sin(A)\sin(B) \le \sin^2\left(\frac{C}{2}\right)$ is a necessary and sufficient condition for the existence of a point D on the segment AB so that CD is the geometrical mean of AD and BD.

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11 We consider the division of a chess board 8×8 in p disjoint rectangles which satisfy the conditions:

a) every rectangle is formed from a number of full squares (not partial) from the 64 and the number of white squares is equal to the number of black squares.

b) the numbers $a_1, \ldots, a_p$ of white squares from p rectangles satisfy $a_1, , \ldots, a_p$. Find the greatest value of p for which there exists such a division and then for that value of p, all the sequences $a_1, \ldots, a_p$ for which we can have such a division.

Moderator says: see `https://artofproblemsolving.com/community/c6h58591`

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12 In a certain language words are formed using an alphabet of three letters. Some words of two or more letters are not allowed, and any two such distinct words are of different lengths. Prove that one can form a word of arbitrary length that does not contain any non-allowed word.
