IMO Shortlist 1975

www.artofproblemsolving.com/community/c3926

by Amir Hossein, orl, Peter

1 There are six ports on a lake. Is it possible to organize a series of routes satisfying the following conditions ?
(i) Every route includes exactly three ports;
(ii) No two routes contain the same three ports;
(iii) The series offers exactly two routes to each tourist who desires to visit two different arbitrary ports.

---

2 We consider two sequences of real numbers $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$. Let $z_1, z_2, \ldots, z_n$ be a permutation of the numbers $y_1, y_2, \ldots, y_n$. Prove that $\sum\limits_{i=1}^{n} (x_i - y_i)^2 \leq \sum\limits_{i=1}^{n} (x_i - z_i)^2$.

---

3 Find the integer represented by $\left[\sum_{n=1}^{10^9} n^{-2/3} \right]$. Here $[x]$ denotes the greatest integer less than or equal to x.

---

4 Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of real numbers such that $0 \leq a_n \leq 1$ and $a_n - 2a_{n+1} + a_{n+2} \geq 0$ for $n = 1, 2, 3, \ldots$. Prove that

$$0 \leq (n+1)(a_n - a_{n+1}) \leq 2 \qquad \text{for } n = 1, 2, 3, \ldots$$

---

5 Let M be the set of all positive integers that do not contain the digit 9 (base 10). If $x_1, \ldots, x_n$ are arbitrary but distinct elements in M, prove that

$$\sum_{j=1}^{n} \frac{1}{x_j} < 80.$$

---

6 When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)

---

7 Prove that from $x + y = 1$ $(x, y \in \mathbb{R})$ it follows that

$$x^{m+1} \sum_{j=0}^{n} \binom{m+j}{j} y^j + y^{n+1} \sum_{i=0}^{m} \binom{n+i}{i} x^i = 1 \qquad (m, n = 0, 1, 2, \ldots).$$

---

8 In the plane of a triangle ABC, in its exterior, we draw the triangles ABR, BCP, CAQ so that $\angle PBC = \angle CAQ = 45^\circ$, $\angle BCP = \angle QCA = 30^\circ$, $\angle ABR = \angle RAB = 15^\circ$.

Prove that

a.) $\angle QRP = 90\,^\circ$, and

b.) $QR = RP$.

---

9 Let $f(x)$ be a continuous function defined on the closed interval $0 \leq x \leq 1$. Let $G(f)$ denote the graph of $f(x) : G(f) = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq 1, y = f(x)\}$. Let $G_a(f)$ denote the graph of the translated function $f(x-a)$ (translated over a distance a), defined by $G_a(f) = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq a+1, y = f(x-a)\}$. Is it possible to find for every $a,\ 0 < a < 1$, a continuous function $f(x)$, defined on $0 \leq x \leq 1$, such that $f(0) = f(1) = 0$ and $G(f)$ and $G_a(f)$ are disjoint point sets ?

---

10 Determine the polynomials P of two variables so that:

a.) for any real numbers t, x, y we have $P(tx, ty) = t^n P(x, y)$ where n is a positive integer, the same for all t, x, y;

b.) for any real numbers a, b, c we have $P(a+b, c) + P(b+c, a) + P(c+a, b) = 0$;

c.) $P(1, 0) = 1$.

---

11 Let $a_1, \ldots, a_n$ be an infinite sequence of strictly positive integers, so that $a_k < a_{k+1}$ for any k. Prove that there exists an infinity of terms a_m, which can be written like $a_m = x \cdot a_p + y \cdot a_q$ with x, y strictly positive integers and $p \neq q$.

---

12 Consider on the first quadrant of the trigonometric circle the arcs $AM_1 = x_1, AM_2 = x_2, AM_3 = x_3, \ldots, AM_v = x_v$, such that $x_1 < x_2 < x_3 < \cdots < x_v$. Prove that

$$\sum_{i=0}^{v-1} \sin 2x_i - \sum_{i=0}^{v-1} \sin(x_i - x_{i+1}) < \frac{\pi}{2} + \sum_{i=0}^{v-1} \sin(x_i + x_{i+1})$$

---

13 Let $A_0, A_1, \ldots, A_n$ be points in a plane such that
(i) $A_0A_1 \leq \frac{1}{2}A_1A_2 \leq \cdots \leq \frac{1}{2^{n-1}}A_{n-1}A_n$ and
(ii) $0 < \measuredangle A_0A_1A_2 < \measuredangle A_1A_2A_3 < \cdots < \measuredangle A_{n-2}A_{n-1}A_n < 180^\circ$,
where all these angles have the same orientation. Prove that the segments A_kA_{k+1}, A_mA_{m+1} do not intersect for each k and n such that $0 \leq k \leq m-2 < n-2$.

---

14 Let $x_0 = 5$ and $x_{n+1} = x_n + \frac{1}{x_n}$ $(n = 0, 1, 2, \ldots)$. Prove that

$$45 < x_{1000} < 45.1.$$

15 Can there be drawn on a circle of radius 1 a number of 1975 distinct points, so that the distance (measured on the chord) between any two points (from the considered points) is a rational number?