IMO Shortlist 1976

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1 Let ABC be a triangle with bisectors AA_1, BB_1, CC_1 ($A_1 \in BC$, etc.) and M their common point. Consider the triangles $MB_1A, MC_1A, MC_1B, MA_1B, MA_1C, MB_1C$, and their inscribed circles. Prove that if four of these six inscribed circles have equal radii, then $AB = BC = CA$.

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2 Let $a_0, a_1, \dots, a_n, a_{n+1}$ be a sequence of real numbers satisfying the following conditions:

$$a_0 = a_{n+1} = 0,$$
$$|a_{k-1} - 2a_k + a_{k+1}| \leq 1 \quad (k = 1, 2, \dots, n).$$

Prove that $|a_k| \leq \frac{k(n+1-k)}{2} \quad (k = 0, 1, \dots, n+1)$.

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3 In a convex quadrilateral (in the plane) with the area of 32 cm^2 the sum of two opposite sides and a diagonal is 16 cm. Determine all the possible values that the other diagonal can have.

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4 A sequence (u_n) is defined by

$$u_0 = 2 \quad u_1 = \frac{5}{2}, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1 \quad \text{for } n = 1, \dots$$

Prove that for any positive integer n we have

$$[u_n] = 2^{\frac{(2^n - (-1)^n)}{3}}$$

(where [x] denotes the smallest integer $\leq$ x).

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5 We consider the following system
with $q = 2p$:

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1q}x_q = 0, \\ a_{21}x_1 + \ldots + a_{2q}x_q = 0, \\ \ldots, \\ a_{p1}x_1 + \ldots + a_{pq}x_q = 0, \end{aligned}$$

in which every coefficient is an element from the set $\{-1, 0, 1\}$. Prove that there exists a solution $x_1, \dots, x_q$ for the system with the properties:

a.) all $x_j, j = 1, \dots, q$ are integers;

b.) there exists at least one j for which $x_j \neq 0$;

c.) $|x_j| \leq q$ for any $j = 1, \ldots, q.$

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6 A box whose shape is a parallelepiped can be completely filled with cubes of side 1. If we put in it the maximum possible number of cubes, each of volume 2, with the sides parallel to those of the box, then exactly 40 percent of the volume of the box is occupied. Determine the possible dimensions of the box.

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7 Let $I = (0, 1]$ be the unit interval of the real line. For a given number $a \in (0, 1)$ we define a map $T : I \to I$ by the formula
if

$$T(x,y) = \begin{cases} x + (1-a), & \text{if } 0 < x \leq a, \\ \\ x - a, & \text{if } a < x \leq 1. \end{cases}$$

Show that for every interval $J \subset I$ there exists an integer $n > 0$ such that $T^n(J) \cap J \neq \emptyset$.

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8 Let P be a polynomial with real coefficients such that $P(x) > 0$ if $x > 0$. Prove that there exist polynomials Q and R with nonnegative coefficients such that $P(x) = \frac{Q(x)}{R(x)}$ if $x > 0$.

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9 Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for j= $2, \ldots$ Prove that for any positive integer n the roots of the equation $P_n(x) = x$ are all real and distinct.

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10 Determine the greatest number, who is the product of some positive integers, and the sum of these numbers is $1976.$

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11 Prove that 5^n has a block of 1976 consecutive $0's$ in its decimal representation.

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12 The polynomial $1976(x + x^2 + \cdots + x^n)$ is decomposed into a sum of polynomials of the form $a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_1, a_2, \ldots, a_n$ are distinct positive integers not greater than n. Find all values of n for which such a decomposition is possible.
