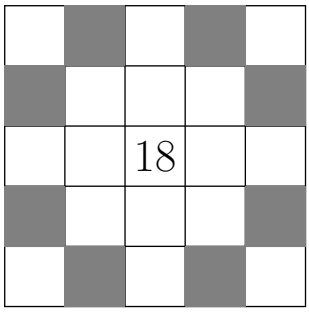
The problems from the CCA Math Bonanza held on 1/14/2017

www.artofproblemsolving.com/community/c392783

by trumpeter

–	Individual Round
I1	Find the integer n such that $6! \times 7! = n!$. *2017 CCA Math Bonanza Individual Round#1*
I2	A rectangle is inscribed in a circle of area 32π and the area of the rectangle is 34. Find its perimeter. *2017 CCA Math Bonanza Individual Round#2*
I3	A sequence starts with 2017 as its first term and each subsequent term is the sum of cubes of the digits in the previous number. What is the 2017th term of this sequence? *2017 CCA Math Bonanza Individual Round#3*
I4	Cole is trying to solve the *Collatz conjecture*. She decides to make a model with a piece of wood with a hole for every natural number. For every even number there is a rope from n to $\frac{n}{2}$ and for every odd number there is a rope from n to $3n+1$. She wants to bring her model to a convention but in order to do that she needs to cut off the part containing the first 240 holes. How many ropes did she break? *2017 CCA Math Bonanza Individual Round#4*
I5	In the *magic square* below, every integer from 1 to 25 can be filled in such that the sum in every row, column, and long diagonal is the same. Given that the number in the center square is 18, what is the sum of the entries in the shaded squares?  *2017 CCA Math Bonanza Individual Round#5*

I6 Determine the largest prime which divides both $2^{24}-1$ and $2^{16}-1$.

2017 CCA Math Bonanza Individual Round#6

I7 Ari the Archer is shooting at an abnormal target. The target consists of 100 concentric rings, each of width 1, so that the total radius of the target is 100. The point value of a given ring of the target is equal to its area (so getting a bull's eye would be worth π points, but hitting on the outer ring would give 199π points). Given that Ari hits any point on the target uniformly at random, what is his expected score?

2017 CCA Math Bonanza Individual Round#7

I8 Let $a_1, a_2, \ldots, a_{18}$ be a list of prime numbers such that $\frac{1}{64} \times a_1 \times a_2 \times \cdots \times a_{18}$ is one million. Determine the sum of all positive integers n such that

$$\sum_{i=1}^{18} \frac{1}{\log_{a_i} n}$$

is a positive integer.

2017 CCA Math Bonanza Individual Round#8

I9 Magic Mark performs a magic trick using a standard 52-card deck except the suits are erased from cards (so that there are 4 identical cards of each rank). He randomly takes 13 cards and uses those to perform his trick. He lets you randomly pick a card from those 13, memorize it, and put it back in the pile of 13 cards. He then shuffles the 13 and takes out a card randomly. If he picks a card identical to yours, the trick is successful. What is probability that the trick is successful?

2017 CCA Math Bonanza Individual Round#9

I10 Find the sum of the two smallest possible values of x° (in degrees) that satisfy the following equation if x is greater than 2017°:

$$\cos^5 9x + \cos^5 x = 32\cos^5 5x \cos^5 4x + 5\cos^2 9x \cos^2 x \left(\cos 9x + \cos x\right).$$

2017 CCA Math Bonanza Individual Round#10

I11 $480\ 1$ cm unit cubes are used to build a block measuring 6 cm by 8 cm by 10 cm. A tiny ant then chews his way in a straight line from one vertex of the block to the furthest vertex. How many cubes does the ant pass through? The ant is so tiny that he does not "pass through" cubes if he is merely passing through where their edges or vertices meet.

2017 CCA Math Bonanza Individual Round#11

I12 Let $a_1, a_2, \ldots, a_{2017}$ be the 2017 distinct complex numbers which satisfy $a_i^{2017} = a_i + 1$ for $i = 1, 2, \ldots, 2017$. Compute

$$\sum_{i=1}^{2017} \frac{a_i}{a_i^2 + 1}.$$

2017 CCA Math Bonanza Individual Round#12

I13 Toner Drum and Celery Hilton are both running for president. A total of 129 million people cast their vote in a random order, with exactly 63 million and 66 million voting for Toner Drum and Celery Hilton, respectively. The Combinatorial News Network displays the face of the leading candidate on the front page of their website. If the two candidates are tied, both faces are displayed. What is the probability that Toner Drum's face is never displayed on the front page?

2017 CCA Math Bonanza Individual Round#13

I14 Find a pair (x, y) of positive integers $x < y$ such that

$$37^2 + 46^2 + 49^2 - 20^2 - 17^2 = x^2 + y^2.$$

Note: there may be several answers; just provide one of them.

[i]2017 CCA Math Bonanza Individual Round#14

I15 Let ABC, $AB < AC$ be an acute triangle inscribed in circle Γ with center O. The altitude from A to BC intersects Γ again at A_1. OA_1 intersects BC at A_2 Similarly define B_1, B_2, C_1, and C_2. Then $B_2C_2 = 2\sqrt{2}$. If B_2C_2 intersects AA_2 at X and BC at Y, then $XB_2 = 2$ and $YB_2 = k$. Find k^2.

2017 CCA Math Bonanza Individual Round#15

– Team Round

T1 Given that $9 \times 10 \times 11 \times \cdots \times 15 = 32432400$, what is $1 \times 3 \times 5 \times \cdots \times 15$?

2017 CCA Math Bonanza Team Round#1

T2 A square of side length s is inscribed in circle C_1 and circumscribed about circle C_2. The area of the region in C_1 but outside C_2 is 25π. What is s?

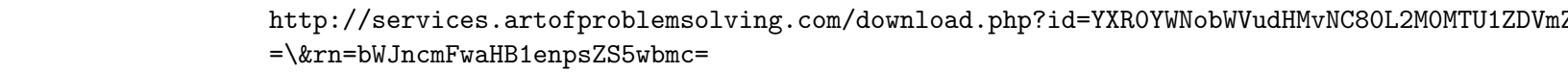
2017 CCA Math Bonanza Team Round#2

T3 The operation $*$ is defined by $a * b = a + b + ab$, where a and b are real numbers. Find the value of

$$\frac{1}{2} * \left(\frac{1}{3} * \left(\cdots * \left(\frac{1}{9} * \left(\frac{1}{10} * \frac{1}{11}\right)\right)\right)\right).$$

2017 CCA Math Bonanza Team Round#3

T4 The 12 islands of the Bonanza archipelago are labeled $A, B, C, \ldots, K, L$. Some of the islands are connected by bridges, as indicated in the diagram below. Tristan wants be able to walk from island to island crossing each bridge exactly once (he doesn't care if he visits a given island more than once, or whether he starts and ends on the same island). Submit a pair of unconnected islands such that if they are connected by a bridge, Tristan can accomplish his goal.



2017 CCA Math Bonanza Team Round#4

T5 Twelve people go to a party. First, everybody with no friends at the party leave. Then, at the i-th hour, everybody with exactly i friends left at the party leave. After the eleventh hour, what is the maximum number of people left? Note that friendship is mutual.

2017 CCA Math Bonanza Team Round#5

T6 Alice and Bob take a and b candies respectively, where $0 \leq a, b \leq 3$, from a pile of 6 identical candies. They draw the candies one at a time, but one person may draw multiple candies in a row. For example, if $a = 2$ and $b = 3$, a possible order of drawing could be Alice, Bob, Bob, Alice, Bob. In how many ways (considering order of drawing and values of a and b) can this happen?

2017 CCA Math Bonanza Team Round#6

T7 Let $ABCD$ be a convex quadrilateral with $AC = 20$, $BC = 12$ and $BD = 17$. If $\angle CAB = 80^\circ$ and $\angle DBA = 70^\circ$, then find the area of $ABCD$.

2017 CCA Math Bonanza Team Round#7

T8 A group of 25 CCA students decide they want to go to Disneyland, which is 105 miles away. To save some time, they rent a bus with capacity 10 people which can travel up to 60 miles per hour. On the other hand, a student will run up to 9 miles per hour. However, because a complicated plan of getting on and off the bus may be confusing to some students, a student may only board the bus once. What is the least number of minutes it will take for all students to reach Disneyland?

Note: both the bus and students may travel backwards.

2017 CCA Math Bonanza Team Round#8

T9 Aida made three cubes with positive integer side lengths a, b, c. They were too small for her, so she divided them into unit cubes and attempted to construct a cube of side $a + b + c$. Unfortunately, she was 648 blocks off. How many possibilities of the ordered triple (a, b, c) are there?

2017 CCA Math Bonanza Team Round#9

T10 Triangle ABC is acute. Equilateral triangles ABC', $AB'C$, $A'BC$ are constructed externally to ABC. Let BB' and CC' intersect at F. Let CC' intersect AB at C_1 and AA' intersect BC at A_1, and let A_1C_1 intersect AC at D. If $A'F = 23$, $CF = 13$, and $DF = 24$, find BD.

2017 CCA Math Bonanza Team Round#10

– Lightning Round

L1.1 Consider the harmonic sequence $\frac{2017}{4}, \frac{2017}{7}, \frac{2017}{10}, \ldots$, where the reciprocals of the terms of the sequence form an arithmetic sequence. How many terms of this sequence are integers?

2017 CCA Math Bonanza Lightning Round#1.1

L1.2 How many ways are there to rearrange the letters of CCARAVEN?

2017 CCA Math Bonanza Lightning Round#1.2

L1.3 Triangle ABC has points A at $(0, 0)$, B at $(9, 12)$, and C at $(-6, 8)$ in the coordinate plane. Find the length of the angle bisector of $\angle BAC$ from A to where it intersects BC.

2017 CCA Math Bonanza Lightning Round#1.3

L1.4 Wild Bill goes to Las Vejas and takes part in a special lottery called *Reverse Yrettol*. In this lottery, a player may buy a ticket on which he or she may select 5 distinct numbers from $1 - 20$ (inclusive). Then, 5 distinct numbers from $1 - 20$ are drawn at random. A player wins if his or her ticket contains *none* of the numbers which were drawn. If Wild Bill buys a ticket, what is the probability that he will win?

2017 CCA Math Bonanza Lightning Round#1.4

L2.1 Adam and Mada are playing a game of one-on-one basketball, in which participants may take 2-point shots (worth 2 points) or 3-point shots (worth 3 points). Adam makes 10 shots of either value while Mada makes 11 shots of either value. Furthermore, Adam made the same number of 2-point shots as Mada made 3-point shots. At the end of the game, the two basketball players realize that they have the exact same number of points! How many total points were scored in the game?

2017 CCA Math Bonanza Lightning Round#2.1

L2.2 Non-degenerate triangle ABC has $AB = 20$, $AC = 17$, and $BC = n$, an integer. How many possible values of n are there?

2017 CCA Math Bonanza Lightning Round#2.2

L2.3 Jack is jumping on the number line. He first jumps one unit and every jump after that he jumps one unit more than the previous jump. What is the least amount of jumps it takes to reach

exactly 19999 from his starting place?

2017 CCA Math Bonanza Lightning Round#2.3

L2.4 Define $f(n) = \text{LCM}(1, 2, \ldots, n)$. Determine the smallest positive integer a such that $f(a) = f(a+2)$.

2017 CCA Math Bonanza Lightning Round#2.4

L3.1 Express $2.3\overline{57}$ as a common fraction.

2017 CCA Math Bonanza Lightning Round#3.1

L3.2 Bob is flipping bottles. Each time he flips the bottle, he has a 0.25 probability of landing it. After successfully flipping a bottle, he has a 0.8 probability of landing his next flip. What is the expected value of the number of times he has to flip the bottle in order to flip it twice in a row?

2017 CCA Math Bonanza Lightning Round#3.2

L3.3 An acute triangle ABC has side lenghths a, b, c such that a, b, c forms an arithmetic sequence. Given that the area of triangle ABC is an integer, what is the smallest value of its perimeter?

2017 CCA Math Bonanza Lightning Round#3.3

L3.4 A random walk is a process in which something moves from point to point, and where the direction of movement at each step is randomly chosen. Suppose that a person conducts a random walk on a line: he starts at 0 and each minute randomly moves either 1 unit in the positive direction or 1 unit in the negative direction. What is his expected distance from the origin after 6 moves?

2017 CCA Math Bonanza Lightning Round#3.4

L4.1 Compute

$$\sum_{k=0}^{\infty} k \left(\frac{1}{3}\right)^k.$$

2017 CCA Math Bonanza Lightning Round#4.1

L4.2 Find $\arctan(1) + \arctan(2) + \arctan(3)$ in radians.

2017 CCA Math Bonanza Lightning Round#4.2

L4.3 Let $f(x)$ be the greatest prime number at most x. Let $g(x)$ be the least prime number greater than x. Find

$$\sum_{i=2}^{100} \frac{1}{f(i)\, g(i)}.$$

2017 CCA Math Bonanza Lightning Round#4.3

L4.4 Let ABC be an acute triangle. $PQRS$ is a rectangle with P on AB, Q and R on BC, and S on AC such that $PQRS$ has the largest area among all rectangles $TUVW$ with T on AB, U and V on BC, and W on AC. If D is the point on BC such that $AD \perp BC$, then PQ is the harmonic mean of $\frac{AD}{DB}$ and $\frac{AD}{DC}$. What is BC?

Note: The harmonic mean of two numbers a and b is the reciprocal of the arithmetic mean of the reciprocals of a and b.

2017 CCA Math Bonanza Lightning Round#4.4

L5.1 Find $x+y+z$ when

$$a_1x + a_2y + a_3z = a$$
$$b_1x + b_2y + b_3z = b$$
$$c_1x + c_2y + c_3z = c$$

Given that

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) = 9$$
$$a(b_2c_3 - b_3c_2) - a_2(bc_3 - b_3c) + a_3(bc_2 - b_2c) = 17$$
$$a_1(bc_3 - b_3c) - a(b_1c_3 - b_3c_1) + a_3(b_1c - bc_1) = -8$$
$$a_1(b_2c - bc_2) - a_2(b_1c - bc_1) + a(b_1c_2 - b_2c_1) = 7.$$

2017 CCA Math Bonanza Lightning Round#5.1

L5.2 Compute $e^\pi + \pi^e$. If your answer is A and the correct answer is C, then your score on this problem will be $\frac{4}{\pi}\arctan\left(\frac{1}{|C-A|}\right)$ (note that the closer you are to the right answer, the higher your score is).

2017 CCA Math Bonanza Lightning Round#5.2

L5.3 How many ways are there to fill a $3\times3\times6$ rectangular prism with $1\times1\times2$ blocks? Rotations are not distinct. If your answer is A and the correct answer is C, then your score on this problem will be $\max\left(2\left(1-\left|\frac{C-A}{C}\right|\right),0\right)$.

2017 CCA Math Bonanza Lightning Round#5.3

L5.4 In the game of Colonel Blotto, you have 100 troops to distribute among 10 castles. Submit a 10-tuple $(x_1, x_2, \ldots x_{10})$ of nonnegative integers such that $x_1+x_2+\cdots+x_{10} = 100$, where each x_i represent the number of troops you want to send to castle i. Your troop distribution will be matched up against each opponent's and you will win 10 points for each castle that you send

more troops to (if you send the same number, you get 5 points, and if you send fewer, you get none). Your aim is to score the most points possible averaged over all opponents.

For example, if team A submits $(90, 10, 0, \ldots, 0)$, team B submits $(11, 11, 11, 11, 11, 11, 11, 11, 11, 1)$, and team C submits $(10, 10, 10, \ldots 10)$, then team A will win 10 points against team B and 15 points against team C, while team B wins 90 points against team C. Team A averages 12.5 points, team B averages 90 points, and team C averages 47.5 points.

2017 CCA Math Bonanza Lightning Round#5.4

– Tiebreaker Round

TB1 Compute

$$12^3 + 4 \times 56 + 7 \times 8 + 9.$$

2017 CCA Math Bonanza Tiebreaker Round#1

TB2 Let ABC be a triangle. D and E are points on line segments BC and AC, respectively, such that $AD = 60$, $BD = 189$, $CD = 36$, $AE = 40$, and $CE = 50$. What is $AB + DE$?

2017 CCA Math Bonanza Tiebreaker Round#2

TB3 Let $\theta = \frac{2\pi}{2015}$, and suppose the product

$$\prod_{k=0}^{1439} \left(\cos(2^k \theta) - \frac{1}{2} \right)$$

can be expressed in the form $\frac{b}{2^a}$, where a is a non-negative integer and b is an odd integer (not necessarily positive). Find $a + b$.

2017 CCA Math Bonanza Tiebreaker Round#3

TB4 Mr. Vader gave out a multiple choice test, and every question had an answer that was one of A, B, or C. After the test, he curved the test so that everybody got +50 (so a person who got $x\%$ right would get a score of $x + 50$). In the class, a score in the range $[90, \infty)$ gets an A, a score in the range $[80, 90)$ gets a B, and a score in the range $[70, 80)$ gets a C. After the curve, Mr. Vader makes this statement: "Guess A, get an A. Guess B, get a B. Guess C, get a C." That is, answering every question with the answer choice X would give, with the curve, a score receiving a grade of X, where X is one of A, B, C. Luke, a student in Mr. Vader's class, was told ahead of time that there were either 5 or 6 answers as A on the test. Find the sum of all possible values of the number of questions on the test, given this information.

2017 CCA Math Bonanza Tiebreaker Round#4
