IMO Shortlist 1977

www.artofproblemsolving.com/community/c3928

by dondigo, Amir Hossein, orl, enescu

1 Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying following condition:

$$f(n+1) > f(f(n)), \quad \forall n \in \mathbb{N}.$$

---

2 A lattice point in the plane is a point both of whose coordinates are integers. Each lattice point has four neighboring points: upper, lower, left, and right. Let k be a circle with radius $r \geq 2$, that does not pass through any lattice point. An interior boundary point is a lattice point lying inside the circle k that has a neighboring point lying outside k. Similarly, an exterior boundary point is a lattice point lying outside the circle k that has a neighboring point lying inside k. Prove that there are four more exterior boundary points than interior boundary points.

---

3 Let a, b be two natural numbers. When we divide $a^2 + b^2$ by $a + b$, we the the remainder r and the quotient q. Determine all pairs (a, b) for which $q^2 + r = 1977$.

---

4 Describe all closed bounded figures Φ in the plane any two points of which are connectable by a semicircle lying in Φ.

---

5 There are 2^n words of length n over the alphabet $\{0, 1\}$. Prove that the following algorithm generates the sequence $w_0, w_1, \dots, w_{2^n-1}$ of all these words such that any two consecutive words differ in exactly one digit.

(1) $w_0 = 00\dots0$ (n zeros).

(2) Suppose $w_{m-1} = a_1 a_2 \dots a_n, \quad a_i \in \{0, 1\}$. Let $e(m)$ be the exponent of 2 in the representation of n as a product of primes, and let $j = 1 + e(m)$. Replace the digit a_j in the word w_{m-1} by $1 - a_j$. The obtained word is w_m.

---

6 Let n be a positive integer. How many integer solutions $(i, j, k, l),\ 1 \leq i, j, k, l \leq n$, does the following system of inequalities have:

$$1 \leq -j + k + l \leq n$$
$$1 \leq i - k + l \leq n$$
$$1 \leq i - j + l \leq n$$
$$1 \leq i + j - k \leq n\,?$$

---

7 Let a, b, A, B be given reals. We consider the function defined by

$$f(x) = 1 - a \cdot \cos(x) - b \cdot \sin(x) - A \cdot \cos(2x) - B \cdot \sin(2x).$$

Prove that if for any real number x we have $f(x) \geq 0$ then $a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1.$

---

8 Let S be a convex quadrilateral $ABCD$ and O a point inside it. The feet of the perpendiculars from O to AB, BC, CD, DA are A_1, B_1, C_1, D_1 respectively. The feet of the perpendiculars from O to the sides of S_i, the quadrilateral $A_iB_iC_iD_i$, are $A_{i+1}B_{i+1}C_{i+1}D_{i+1}$, where $i = 1, 2, 3$. Prove that S_4 is similar to S.

---

9 For which positive integers n do there exist two polynomials f and g with integer coefficients of n variables $x_1, x_2, \ldots, x_n$ such that the following equality is satisfied:

$$\sum_{i=1}^{n} x_i f(x_1, x_2, \ldots, x_n) = g(x_1^2, x_2^2, \ldots, x_n^2) \ ?$$

---

10 Let n be a given number greater than 2. We consider the set V_n of all the integers of the form $1 + kn$ with $k = 1, 2, \ldots$ A number m from V_n is called indecomposable in V_n if there are not two numbers p and q from V_n so that $m = pq$. Prove that there exist a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Expressions which differ only in order of the elements of V_n will be considered the same.)

---

11 Let n be an integer greater than 1. Define

$$x_1 = n, y_1 = 1, x_{i+1} = \left[\frac{x_i + y_i}{2}\right], y_{i+1} = \left[\frac{n}{x_{i+1}}\right], \qquad \text{for } i = 1, 2, \ldots\ ,$$

where $[z]$ denotes the largest integer less than or equal to z. Prove that

$$\min\{x_1, x_2, \ldots, x_n\} = [\sqrt{n}]$$

---

12 In the interior of a square $ABCD$ we construct the equilateral triangles ABK, BCL, CDM, DAN. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AK, BK, BL, CL, CM, DM, DN, AN$ are the 12 vertices of a regular dodecagon.

---

13 Let B be a set of k sequences each having n terms equal to 1 or -1. The product of two such sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ is defined as $(a_1b_1, a_2b_2, \ldots, a_nb_n)$. Prove that there exists a sequence $(c_1, c_2, \ldots, c_n)$ such that the intersection of B and the set containing all sequences from B multiplied by $(c_1, c_2, \ldots, c_n)$ contains at most $\frac{k^2}{2^n}$ sequences.

---

14 Let E be a finite set of points such that E is not contained in a plane and no three points of E are collinear. Show that at least one of the following alternatives holds:

(i) E contains five points that are vertices of a convex pyramid having no other points in common with E;

(ii) some plane contains exactly three points from E.

---

15 In a finite sequence of real numbers the sum of any seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

---

16 Let E be a set of n points in the plane $(n \geq 3)$ whose coordinates are integers such that any three points from E are vertices of a nondegenerate triangle whose centroid doesnt have both coordinates integers. Determine the maximal n.
