IMO Shortlist 1978

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1 The set $M = \{1, 2, ..., 2n\}$ is partitioned into k nonintersecting subsets $M_1, M_2, \dots, M_k$, where $n \ge k^3 + k$. Prove that there exist even numbers $2j_1, 2j_2, \dots, 2j_{k+1}$ in M that are in one and the same subset M_i $(1 \le i \le k)$ such that the numbers $2j_1 - 1, 2j_2 - 1, \dots, 2j_{k+1} - 1$ are also in one and the same subset $M_j (1 \le j \le k)$.

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2 Two identically oriented equilateral triangles, ABC with center S and $A'B'C$, are given in the plane. We also have $A' \neq S$ and $B' \neq S$. If M is the midpoint of $A'B$ and N the midpoint of AB', prove that the triangles $SB'M$ and $SA'N$ are similar.

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3 Let m and n be positive integers such that $1 \le m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that $m + n$ has its least value.

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4 Let T_1 be a triangle having a, b, c as lengths of its sides and let T_2 be another triangle having u, v, w as lengths of its sides. If P, Q are the areas of the two triangles, prove that

$$16PQ \le a^2(-u^2 + v^2 + w^2) + b^2(u^2 - v^2 + w^2) + c^2(u^2 + v^2 - w^2).$$

When does equality hold?

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5 For every integer $d \ge 1$, let M_d be the set of all positive integers that cannot be written as a sum of an arithmetic progression with difference d, having at least two terms and consisting of positive integers. Let $A = M_1$, $B = M_2 \setminus \{2\}, C = M_3$. Prove that every $c \in C$ may be written in a unique way as $c = ab$ with $a \in A, b \in B$.

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6 Let f be an injective function from $1, 2, 3, \dots$ in itself. Prove that for any n we have: $\sum_{k=1}^{n} f(k)k^{-2} \ge \sum_{k=1}^{n} k^{-1}$.

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7 We consider three distinct half-lines Ox, Oy, Oz in a plane. Prove the existence and uniqueness of three points $A \in Ox, B \in Oy, C \in Oz$ such that the perimeters of the triangles OAB, OBC, OCA are all equal to a given number $2p > 0$.

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8 Let S be the set of all the odd positive integers that are not multiples of 5 and that are less than $30m$, m being an arbitrary positive integer. What is the smallest integer k such that in any subset of k integers from S there must be two different integers, one of which divides the other?

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9 Let $0 < f(1) < f(2) < f(3) < \ldots$ a sequence with all its terms positive. The $n-th$ positive integer which doesn't belong to the sequence is $f(f(n))+1$. Find $f(240)$.

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10 An international society has its members from six different countries. The list of members contain 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

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11 A function $f : I \to \mathbb{R}$, defined on an interval I, is called concave if $f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta)f(y)$ for all $x, y \in I$ and $0 \leq \theta \leq 1$. Assume that the functions $f_1, \ldots, f_n$, having all nonnegative values, are concave. Prove that the function $(f_1 f_2 \cdots f_n)^{1/n}$ is concave.

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12 In a triangle ABC we have $AB = AC$. A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides AB, AC in the points P, respectively Q. Prove that the midpoint of PQ is the center of the inscribed circle of the triangle ABC.

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13 We consider a fixed point P in the interior of a fixed sphere. We construct three segments PA, PB, PC, perpendicular two by two, with the vertexes A, B, C on the sphere. We consider the vertex Q which is opposite to P in the parallelepiped (with right angles) with PA, PB, PC as edges. Find the locus of the point Q when A, B, C take all the positions compatible with our problem.

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14 Prove that it is possible to place $2n(2n+1)$ parallelepipedic (rectangular) pieces of soap of dimensions $1 \times 2 \times (n+1)$ in a cubic box with edge $2n+1$ if and only if n is even or $n = 1$.

Remark. It is assumed that the edges of the pieces of soap are parallel to the edges of the box.

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15 Let p be a prime and $A = \{a_1, \ldots, a_{p-1}\}$ an arbitrary subset of the set of natural numbers such that none of its elements is divisible by p. Let us define a mapping f from $\mathcal{P}(A)$ (the set of all subsets of A) to the set $P = \{0, 1, \ldots, p-1\}$ in the following way:

(i) if $B = \{a_{i_1}, \ldots, a_{i_k}\} \subset A$ and $\sum_{j=1}^{k} a_{i_j} \equiv n \pmod p$, then $f(B) = n$,

(ii) $f(\emptyset) = 0$, $\emptyset$ being the empty set.

Prove that for each $n \in P$ there exists $B \subset A$ such that $f(B) = n$.

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16 Determine all the triples (a, b, c) of positive real numbers such that the system

$$ax + by - cz = 0,$$

$$a\sqrt{1-x^2} + b\sqrt{1-y^2} - c\sqrt{1-z^2} = 0,$$

is compatible in the set of real numbers, and then find all its real solutions.

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17 Prove that for any positive integers x, y, z with $xy - z^2 = 1$ one can find non-negative integers a, b, c, d such that $x = a^2 + b^2, y = c^2 + d^2, z = ac + bd$.
Set $z = (2q)!$ to deduce that for any prime number $p = 4q + 1$, p can be represented as the sum of squares of two integers.