IMO Shortlist 1979

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1 Prove that in the Euclidean plane every regular polygon having an even number of sides can be dissected into lozenges. (A lozenge is a quadrilateral whose four sides are all of equal length).

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2 From a bag containing 5 pairs of socks, each pair a different color, a random sample of 4 single socks is drawn. Any complete pairs in the sample are discarded and replaced by a new pair draw from the bag. The process continues until the bag is empty or there are 4 socks of different colors held outside the bag. What is the probability of the latter alternative?

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3 Find all polynomials $f(x)$ with real coefficients for which

$$f(x)f(2x^2) = f(2x^3 + x).$$

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4 We consider a prism which has the upper and inferior basis the pentagons: $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$. Each of the sides of the two pentagons and the segments A_iB_j with $i, j = 1, \dots, 5$ is colored in red or blue. In every triangle which has all sides colored there exists one red side and one blue side. Prove that all the 10 sides of the two basis are colored in the same color.

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5 Let $n \geq 2$ be an integer. Find the maximal cardinality of a set M of pairs (j, k) of integers, $1 \leq j < k \leq n$, with the following property: If $(j, k) \in M$, then $(k, m) \notin M$ for any m.

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6 Find the real values of p for which the equation

$$\sqrt{2p + 1 - x^2} + \sqrt{3x + p + 4} = \sqrt{x^2 + 9x + 3p + 9}$$

in x has exactly two real distinct roots.($\sqrt{t}$ means the positive square root of t).

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7 If p and q are natural numbers so that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{1318} + \frac{1}{1319},$$

prove that p is divisible with 1979.

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8 For all rational x satisfying $0 \leq x < 1$, the functions f is defined by

$$f(x) = \begin{cases} \frac{f(2x)}{4}, & \text{for } 0 \leq x < \frac{1}{2}, \\ \frac{3}{4} + \frac{f(2x-1)}{4}, & \text{for } \frac{1}{2} \leq x < 1. \end{cases}$$

Given that $x = 0.b_1b_2b_3\cdots$ is the binary representation of x, find, with proof, $f(x)$.

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9 Let A and E be opposite vertices of an octagon. A frog starts at vertex A. From any vertex except E it jumps to one of the two adjacent vertices. When it reaches E it stops. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that:

$$a_{2n-1} = 0, \quad a_{2n} = \frac{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}}{\sqrt{2}}.$$

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10 Show that for any vectors a, b in Euclidean space,

$$|a \times b|^3 \leq \frac{3\sqrt{3}}{8}|a|^2|b|^2|a-b|^2$$

Remark. Here $\times$ denotes the vector product.

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11 Given real numbers $x_1, x_2, \dots, x_n$ $(n \geq 2)$, with $x_i \geq \frac{1}{n}$ $(i = 1, 2, \dots, n)$ and with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, find whether the product $P = x_1x_2x_3\cdots x_n$ has a greatest and/or least value and if so, give these values.

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12 Let R be a set of exactly 6 elements. A set F of subsets of R is called an S-family over R if and only if it satisfies the following three conditions:
(i) For no two sets X, Y in F is $X \subseteq Y$;

(ii) For any three sets X, Y, Z in F, $X \cup Y \cup Z \neq R,$

(iii) $\bigcup_{X \in F} X = R$

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13 Show that $\frac{20}{60} < \sin 20^\circ < \frac{21}{60}$.

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14 Find all bases of logarithms in which a real positive number can be equal to its logarithm or prove that none exist.

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15 Determine all real numbers a for which there exists positive reals $x_1, \dots, x_5$ which satisfy the relations $\sum_{k=1}^5 kx_k = a$, $\sum_{k=1}^5 k^3x_k = a^2$, $\sum_{k=1}^5 k^5x_k = a^3$.

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16 Let K denote the set $\{a, b, c, d, e\}$. F is a collection of 16 different subsets of K, and it is known that any three members of F have at least one element in common. Show that all 16 members of F have exactly one element in common.

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17 Inside an equilateral triangle ABC one constructs points P, Q and R such that

$$\angle QAB = \angle PBA = 15^\circ, \angle RBC = \angle QCB = 20^\circ, \angle PCA = \angle RAC = 25^\circ.$$

Determine the angles of triangle PQR.

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18 Let m positive integers $a_1, \dots, a_m$ be given. Prove that there exist fewer than 2^m positive integers $b_1, \dots, b_n$ such that all sums of distinct b_ks are distinct and all a_i $(i \leq m)$ occur among them.

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19 Consider the sequences $(a_n), (b_n)$ defined by

$$a_1 = 3, \quad b_1 = 100, \quad a_{n+1} = 3^{a_n}, \quad b_{n+1} = 100^{b_n}$$

Find the smallest integer m for which $b_m > a_{100}$.

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20 Given the integer $n > 1$ and the real number $a > 0$ determine the maximum of $\sum_{i=1}^{n-1} x_i x_{i+1}$ taken over all nonnegative numbers x_i with sum a.

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21 Let N be the number of integral solutions of the equation

$$x^2 - y^2 = z^3 - t^3$$

satisfying the condition $0 \leq x, y, z, t \leq 10^6$, and let M be the number of integral solutions of the equation

$$x^2 - y^2 = z^3 - t^3 + 1$$

satisfying the condition $0 \leq x, y, z, t \leq 10^6$. Prove that $N > M$.

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22 Two circles in a plane intersect. A is one of the points of intersection. Starting simultaneously from A two points move with constant speed, each travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that the two points are always equidistant from P.

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23 Find all natural numbers n for which $2^8 + 2^{11} + 2^n$ is a perfect square.

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24 A circle C with center O on base BC of an isosceles triangle ABC is tangent to the equal sides AB, AC. If point P on AB and point Q on AC are selected such that $PB \times CQ = (\frac{BC}{2})^2$, prove that line segment PQ is tangent to circle C, and prove the converse.

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25 We consider a point P in a plane p and a point $Q \notin p$. Determine all the points R from p for which

$$\frac{QP + PR}{QR}$$

is maximum.

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26 Prove that the functional equations

$$f(x+y) = f(x) + f(y),$$

$$\text{and} \qquad f(x+y+xy) = f(x) + f(y) + f(xy) \quad (x, y \in \mathbb{R})$$

are equivalent.
