IMO Shortlist 1980

www.artofproblemsolving.com/community/c3931

by orl

1 Let α, β and γ denote the angles of the triangle ABC. The perpendicular bisector of AB intersects BC at the point X, the perpendicular bisector of AC intersects it at Y. Prove that $\tan(\beta) \cdot \tan(\gamma) = 3$ implies $BC = XY$ (or in other words: Prove that a sufficient condition for $BC = XY$ is $\tan(\beta) \cdot \tan(\gamma) = 3$). Show that this condition is not necessary, and give a necessary and sufficient condition for $BC = XY$.

---

2 Define the numbers $a_0, a_1, \dots, a_n$ in the following way:

$$a_0 = \frac{1}{2}, \quad a_{k+1} = a_k + \frac{a_k^2}{n} \quad (n > 1, k = 0, 1, \dots, n-1).$$

Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

---

3 Prove that the equation

$$x^n + 1 = y^{n+1},$$

where n is a positive integer not smaller then 2, has no positive integer solutions in x and y for which x and $n+1$ are relatively prime.

---

4 Determine all positive integers n such that the following statement holds: If a convex polygon with with $2n$ sides $A_1A_2 \dots A_{2n}$ is inscribed in a circle and $n-1$ of its n pairs of opposite sides are parallel, which means if the pairs of opposite sides

$$(A_1A_2, A_{n+1}A_{n+2}), (A_2A_3, A_{n+2}A_{n+3}), \dots, (A_{n-1}A_n, A_{2n-1}A_{2n})$$

are parallel, then the sides

$$A_nA_{n+1}, A_{2n}A_1$$

are parallel as well.

---

5 In a rectangular coordinate system we call a horizontal line parallel to the x -axis triangular if it intersects the curve with equation

$$y = x^4 + px^3 + qx^2 + rx + s$$

in the points A, B, C and D (from left to right) such that the segments AB, AC and AD are the sides of a triangle. Prove that the lines parallel to the x - axis intersecting the curve in four distinct points are all triangular or none of them is triangular.

---

6 Find the digits left and right of the decimal point in the decimal form of the number

$$(\sqrt{2}+\sqrt{3})^{1980}.$$

---

7 The function f is defined on the set $\mathbb{Q}$ of all rational numbers and has values in $\mathbb{Q}$. It satisfies the conditions $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x+y) + 1$ for all $x, y \in \mathbb{Q}$. Determine f.

---

8 Three points A, B, C are such that $B \in]AC[$. On the side of AC we draw the three semicircles with diameters $[AB], [BC]$ and $[AC]$. The common interior tangent at B to the first two semi-circles meets the third circle in E. Let U and V be the points of contact of the common exterior tangent to the first two semi-circles. Denote the area of the triangle ABC as $S(ABC)$. Evaluate the ratio $R = \frac{S(EUV)}{S(EAC)}$ as a function of $r_1 = \frac{AB}{2}$ and $r_2 = \frac{BC}{2}$.

---

9 Let p be a prime number. Prove that there is no number divisible by p in the $n-th$ row of Pascal's triangle if and only if n can be represented in the form $n = p^s q - 1$, where s and q are integers with $s \geq 0, 0 < q < p$.

---

10 Two circles C_1 and C_2 are (externally or internally) tangent at a point P. The straight line D is tangent at A to one of the circles and cuts the other circle at the points B and C. Prove that the straight line PA is an interior or exterior bisector of the angle $\angle BPC$.

---

11 Ten gamblers started playing with the same amount of money. Each turn they cast (threw) five dice. At each stage the gambler who had thrown paid to each of his 9 opponents $\frac{1}{n}$ times the amount which that opponent owned at that moment. They threw and paid one after the other. At the 10th round (i.e. when each gambler has cast the five dice once), the dice showed a total of 12, and after payment it turned out that every player had exactly the same sum as he had at the beginning. Is it possible to determine the total shown by the dice at the nine former rounds ?

---

12 Find all pairs of solutions (x, y):

$$x^3 + x^2y + xy^2 + y^3 = 8(x^2 + xy + y^2 + 1).$$

---

13 Given three infinite arithmetic progressions of natural numbers such that each of the numbers 1,2,3,4,5,6,7 and 8 belongs to at least one of them, prove that the number 1980 also belongs to at least one of them.

---

14 Let $\{x_n\}$ be a sequence of natural numbers such that

$$(a) 1 = x_1 < x_2 < x_3 < \dots; \quad (b) x_{2n+1} \leq 2n \quad \forall n.$$

Prove that, for every natural number k, there exist terms x_r and x_s such that $x_r - x_s = k$.

---

15 Prove that the sum of the six angles subtended at an interior point of a tetrahedron by its six edges is greater than 540.

---

16 Prove that $\sum \frac{1}{i_1 i_2 \dots i_k} = n$ is taken over all non-empty subsets $\{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, n\}$. (The k is not fixed, so we are summing over all the $2^n - 1$ possible nonempty subsets.)

---

17 Let $A_1A_2A_3$ be a triangle and, for $1 \leq i \leq 3$, let B_i be an interior point of edge opposite A_i. Prove that the perpendicular bisectors of A_iB_i for $1 \leq i \leq 3$ are not concurrent.

---

18 Given a sequence $\{a_n\}$ of real numbers such that $|a_{k+m} - a_k - a_m| \leq 1$ for all positive integers k and m, prove that, for all positive integers p and q,

$$|\frac{a_p}{p} - \frac{a_q}{q}| < \frac{1}{p} + \frac{1}{q}.$$

---

19 Find the greatest natural number n such there exist natural numbers $x_1, x_2, \dots, x_n$ and natural $a_1 < a_2 < \dots < a_{n-1}$ satisfying the following equations for $i = 1, 2, \dots, n-1$:

$$x_1 x_2 \dots x_n = 1980 \quad \text{and} \quad x_i + \frac{1980}{x_i} = a_i.$$

---

20 Let S be a set of 1980 points in the plane such that the distance between every pair of them is at least 1. Prove that S has a subset of 220 points such that the distance between every pair of them is at least $\sqrt{3}$.

---

21 Let AB be a diameter of a circle; let t_1 and t_2 be the tangents at A and B, respectively; let C be any point other than A on t_1; and let $D_1D_2.E_1E_2$ be arcs on the circle determined by two lines through C. Prove that the lines AD_1 and AD_2 determine a segment on t_2 equal in length to that of the segment on t_2 determined by AE_1 and AE_2.
