IMO Shortlist 1981

www.artofproblemsolving.com/community/c3932

by orl, Boy Soprano II, Amir Hossein, yetti

1 **a.)** For which $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n-1$ numbers?

b.) For which $n > 2$ is there exactly one set having this property?

---

2 A sphere S is tangent to the edges AB, BC, CD, DA of a tetrahedron $ABCD$ at the points E, F, G, H respectively. The points E, F, G, H are the vertices of a square. Prove that if the sphere is tangent to the edge AC, then it is also tangent to the edge BD.

---

3 Find the minimum value of

$$\max(a+b+c, b+c+d, c+d+e, d+e+f, e+f+g)$$

subject to the constraints

(i) $a, b, c, d, e, f, g \geq 0,$

(ii)$a+b+c+d+e+f+g = 1.$

---

4 Let $\{fn\}$ be the Fibonacci sequence $\{1, 1, 2, 3, 5, \dots.\}$.

(a) Find all pairs (a, b) of real numbers such that for each n, $af_n + bf_{n+1}$ is a member of the sequence.

(b) Find all pairs (u, v) of positive real numbers such that for each n, $uf_n^2 + vf_{n+1}^2$ is a member of the sequence.

---

5 A cube is assembled with 27 white cubes. The larger cube is then painted black on the outside and disassembled. A blind man reassembles it. What is the probability that the cube is now completely black on the outside? Give an approximation of the size of your answer.

---

6 Let $P(z)$ and $Q(z)$ be complex-variable polynomials, with degree not less than 1. Let

$$P_k = \{z \in \mathbb{C} | P(z) = k\}, Q_k = \{z \in \mathbb{C} | Q(z) = k\}.$$

Let also $P_0 = Q_0$ and $P_1 = Q_1$. Prove that $P(z) \equiv Q(z)$.

---

7 The function $f(x, y)$ satisfies: $f(0, y) = y+1, f(x+1, 0) = f(x, 1), f(x+1, y+1) = f(x, f(x+1, y))$ for all non-negative integers x, y. Find $f(4, 1981)$.

---

8 Take r such that $1 \le r \le n$, and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each subset has a smallest element. Let $F(n, r)$ be the arithmetic mean of these smallest elements. Prove that:

$$F(n, r) = \frac{n+1}{r+1}.$$

---

9 A sequence (a_n) is defined by means of the recursion

$$a_1 = 1, a_{n+1} = \frac{1 + 4a_n + \sqrt{1 + 24a_n}}{16}.$$

Find an explicit formula for a_n.

---

10 Determine the smallest natural number n having the following property: For every integer $p, p \ge n$, it is possible to subdivide (partition) a given square into p squares (not necessarily equal).

---

11 On a semicircle with unit radius four consecutive chords AB, BC, CD, DE with lengths a, b, c, d, respectively, are given. Prove that

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4.$$

---

12 Determine the maximum value of m^2+n^2, where m and n are integers in the range $1, 2, \ldots, 1981$ satisfying $(n^2 - mn - m^2)^2 = 1$.

---

13 Let P be a polynomial of degree n satisfying

$$P(k) = \binom{n+1}{k}^{-1} \qquad \text{for } k = 0, 1, ..., n.$$

Determine $P(n+1)$.

---

14 Prove that a convex pentagon (a five-sided polygon) $ABCDE$ with equal sides and for which the interior angles satisfy the condition $\angle A \ge \angle B \ge \angle C \ge \angle D \ge \angle E$ is a regular pentagon.

---

15 Consider a variable point P inside a given triangle ABC. Let D, E, F be the feet of the perpendiculars from the point P to the lines BC, CA, AB, respectively. Find all points P which minimize the sum

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}.$$

---

16 A sequence of real numbers $u_1, u_2, u_3, \dots$ is determined by u_1 and the following recurrence relation for $n \geq 1$:

$$4u_{n+1} = \sqrt[3]{64u_n + 15.}$$

Describe, with proof, the behavior of u_n as $n \to \infty$.

---

17 Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.

---

18 Several equal spherical planets are given in outer space. On the surface of each planet there is a set of points that is invisible from any of the remaining planets. Prove that the sum of the areas of all these sets is equal to the area of the surface of one planet.

---

19 A finite set of unit circles is given in a plane such that the area of their union U is S. Prove that there exists a subset of mutually disjoint circles such that the area of their union is greater that $\frac{2S}{9}$.
