VMO 2017

www.artofproblemsolving.com/community/c393299

by v_Enhance, rterte, Strong_TTM, Navi_Makerloff

Day 1 4 problems

---

1 Given $a \in \mathbb{R}$ and a sequence (u_n) defined by

$$\begin{cases} u_1 = a \\ u_{n+1} = \frac{1}{2} + \sqrt{\frac{2n+3}{n+1}u_n + \frac{1}{4}} \quad \forall n \in \mathbb{N}^* \end{cases}$$

a) Prove that (u_n) is convergent sequence when $a = 5$ and find the limit of the sequence in that case

b) Find all a such that the sequence (u_n) is exist and is convergent.

---

2 Is there an integer coefficients polynomial $P(x)$ satisfying

$$\begin{cases} P(1+\sqrt[3]{2}) = 1+\sqrt[3]{2} \\ P(1+\sqrt{5}) = 2+3\sqrt{5} \end{cases}$$

---

3 Given an acute, non isoceles triangle ABC and (O) be its circumcircle, H its orthocenter and E, F are the feet of the altitudes from B and C, respectively. AH intersects (O) at D ($D \neq A$).

a) Let I be the midpoint of AH, EI meets BD at M and FI meets CD at N. Prove that $MN \perp OH$.

b) The lines DE, DF intersect (O) at P, Q respectively ($P \neq D, Q \neq D$). (AEF) meets (O) and AO at R, S respectively ($R \neq A, S \neq A$). Prove that BP, CQ, RS are concurrent.

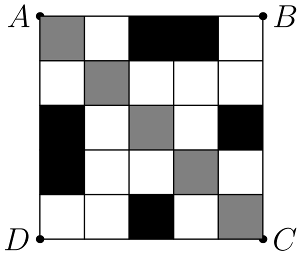
---

4 Given an integer $n > 1$ and a $n \times n$ grid $ABCD$ containing n^2 unit squares, each unit square is colored by one of three colors: Black, white and gray. A coloring is called *symmetry* if each unit square has center on diagonal AC is colored by gray and every couple of unit squares which are symmetry by AC should be both colred by black or white. In each gray square, they label a number 0, in a white square, they will label a positive integer and in a black square, a negative integer. A label will be called k-*balance* (with $k \in \mathbb{Z}^+$) if it satisfies the following requirements:

i) Each pair of unit squares which are symmetry by AC are labelled with the same integer from the closed interval $[-k, k]$

ii) If a row and a column intersectes at a square that is colored by black, then the set of positive integers on that row and the set of positive integers on that column are distinct.If a row and a column intersectes at a square that is colored by white, then the set of negative integers on that row and the set of negative integers on that column are distinct.

a) For $n=5$, find the minimum value of k such that there is a k-balance label for the following grid



b) Let $n=2017$. Find the least value of k such that there is always a k-balance label for a symmetry coloring.

Day 2 3 problems

1 Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying relation :

$$f(xf(y)-f(x))=2f(x)+xy$$

$\forall x,y\in\mathbb{R}$

2 Prove that
a)$\sum_{k=1}^{1008} kC_{2017}^{k}\equiv 0$ (mod 2017^2)

b)$\sum_{k=1}^{504}\left(-1\right)^{k}C_{2017}^{k}\equiv 3\left(2^{2016}-1\right)$ (mod 2017^2)

3 Given an acute triangle ABC and (O) be its circumcircle. Let G be the point on arc BC that doesn't contain O of the circumcircle (I) of triangle OBC. The circumcircle of ABG intersects AC at E and circumcircle of ACG intersects AB at F ($E\neq A, F\neq A$).

a) Let K be the intersection of BE and CF. Prove that AK, BC, OG are concurrent.

b) Let D be a point on arc BOC (arc BC containing O) of (I). GB meets CD at M , GC meets BD at N. Assume that MN intersects (O) at P nad Q. Prove that when G moves on the arc BC that doesn't contain O of (I), the circumcircle (GPQ) always passes through two fixed points.