IMO Shortlist 1982

www.artofproblemsolving.com/community/c3933

by orl, Amir Hossein, grobber

1 The function $f(n)$ is defined on the positive integers and takes non-negative integer values. $f(2) = 0, f(3) > 0, f(9999) = 3333$ and for all m, n :

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1.$$

Determine $f(1982)$.

---

2 Let K be a convex polygon in the plane and suppose that K is positioned in the coordinate system in such a way that

$$\text{area } (K \cap Q_i) = \frac{1}{4}\text{area } K \ (i = 1, 2, 3, 4,),$$

where the Q_i denote the quadrants of the plane. Prove that if K contains no nonzero lattice point, then the area of K is less than 4.

---

3 Consider infinite sequences $\{x_n\}$ of positive reals such that $x_0 = 1$ and $x_0 \geq x_1 \geq x_2 \geq \ldots$.

a) Prove that for every such sequence there is an $n \geq 1$ such that:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

b) Find such a sequence such that for all n:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} < 4.$$

---

4 Determine all real values of the parameter a for which the equation

$$16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$$

has exactly four distinct real roots that form a geometric progression.

---

5 The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by inner points M and N respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M and N are collinear.

---

6 Let S be a square with sides length 100. Let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ with $A_0 = A_n$. Suppose that for every point P on the boundary of S there is a point of L at a distance from P no greater than $\frac{1}{2}$. Prove that there are two points X and Y of L such that the distance between X and Y is not greater than 1 and the length of the part of L which lies between X and Y is not smaller than 198.

---

7 Let $p(x)$ be a cubic polynomial with integer coefficients with leading coefficient 1 and with one of its roots equal to the product of the other two. Show that $2p(-1)$ is a multiple of $p(1) + p(-1) - 2(1 + p(0))$.

---

8 A convex, closed figure lies inside a given circle. The figure is seen from every point of the circumference at a right angle (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the center of the circle is a center of symmetry of the figure.

---

9 Let ABC be a triangle, and let P be a point inside it such that $\angle PAC = \angle PBC$. The perpendiculars from P to BC and CA meet these lines at L and M, respectively, and D is the midpoint of AB. Prove that $DL = DM$.

---

10 A box contains p white balls and q black balls. Beside the box there is a pile of black balls. Two balls are taken out of the box. If they have the same color, a black ball from the pile is put into the box. If they have different colors, the white ball is put back into the box. This procedure is repeated until the last two balls are removed from the box and one last ball is put in. What is the probability that this last ball is white?

---

11 **(a)** Find the rearrangement $\{a_1, \ldots, a_n\}$ of $\{1, 2, \ldots, n\}$ that maximizes

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 = Q.$$

(b) Find the rearrangement that minimizes Q.

---

12 Four distinct circles C, C_1, C_2, C3 and a line L are given in the plane such that C and L are disjoint and each of the circles C_1, C_2, C_3 touches the other two, as well as C and L. Assuming the radius of C to be 1, determine the distance between its center and L.

---

13 A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with the side a_i lying opposite to the vertex A_i. Let M_i be the midpoint of the side a_i, and let T_i be the point where the inscribed circle of triangle $A_1A_2A_3$ touches the side a_i. Denote by S_i the reflection of the point T_i in the interior angle bisector of the angle A_i. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

---

14 Let $ABCD$ be a convex plane quadrilateral and let A_1 denote the circumcenter of $\triangle BCD$. Define B_1, C_1, D_1 in a corresponding way.

(a) Prove that either all of A_1, B_1, C_1, D_1 coincide in one point, or they are all distinct. Assuming the latter case, show that A_1, C1 are on opposite sides of the line B_1D_1, and similarly,B_1, D_1 are on opposite sides of the line A_1C_1. (This establishes the convexity of the quadrilateral $A_1B_1C_1D_1$.)

(b) Denote by A_2 the circumcenter of $B_1C_1D_1$, and define B_2, C_2, D_2 in an analogous way. Show that the quadrilateral $A_2B_2C_2D_2$ is similar to the quadrilateral $ABCD.$

---

15 Show that

$$\frac{1-s^a}{1-s} \leq (1+s)^{a-1}$$

holds for every $1 \neq s > 0$ real and $0 < a \leq 1$ rational.

---

16 Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers x, y, then it has at least three such solutions. Show that the equation has no solutions in integers for $n = 2891$.

---

17 The right triangles ABC and AB_1C_1 are similar and have opposite orientation. The right angles are at C and C_1, and we also have $\angle CAB = \angle C_1AB_1$. Let M be the point of intersection of the lines BC_1 and B_1C. Prove that if the lines AM and CC_1 exist, they are perpendicular.

---

18 Let O be a point of three-dimensional space and let l_1, l_2, l_3 be mutually perpendicular straight lines passing through O. Let S denote the sphere with center O and radius R, and for every point M of S, let S_M denote the sphere with center M and radius R. We denote by P_1, P_2, P_3 the intersection of S_M with the straight lines l_1, l_2, l_3, respectively, where we put $P_i \neq O$ if l_i meets S_M at two distinct points and $P_i = O$ otherwise ($i = 1, 2, 3$). What is the set of centers of gravity of the (possibly degenerate) triangles $P_1P_2P_3$ as M runs through the points of S?

---

19 Let M be the set of real numbers of the form $\frac{m+n}{\sqrt{m^2+n^2}}$, where m and n are positive integers. Prove that for every pair $x \in M, y \in M$ with $x < y$, there exists an element $z \in M$ such that $x < z < y$.

---

20 Let $ABCD$ be a convex quadrilateral and draw regular triangles ABM, CDP, BCN, ADQ, the first two outward and the other two inward. Prove that $MN = AC$. What can be said about the quadrilateral $MNPQ$?
