IMO Shortlist 1983

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1 The localities $P_1, P_2, \dots, P_{1983}$ are served by ten international airlines $A_1, A_2, \dots, A_{10}$. It is noticed that there is direct service (without stops) between any two of these localities and that all airline schedules offer round-trip flights. Prove that at least one of the airlines can offer a round trip with an odd number of landings.

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2 Let n be a positive integer. Let $\sigma(n)$ be the sum of the natural divisors d of n (including 1 and n). We say that an integer $m \geq 1$ is *superabundant* (P.Erdos, 1944) if $\forall k \in \{1, 2, \dots, m-1\}$, $\frac{\sigma(m)}{m} > \frac{\sigma(k)}{k}$.
Prove that there exists an infinity of *superabundant* numbers.

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3 Let ABC be an equilateral triangle and $\mathcal{E}$ the set of all points contained in the three segments AB, BC, and CA (including A, B, and C). Determine whether, for every partition of $\mathcal{E}$ into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.

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4 On the sides of the triangle ABC, three similar isosceles triangles ABP $(AP = PB)$, AQC $(AQ = QC)$, and BRC $(BR = RC)$ are constructed. The first two are constructed externally to the triangle ABC, but the third is placed in the same half-plane determined by the line BC as the triangle ABC. Prove that $APRQ$ is a parallelogram.

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5 Consider the set of all strictly decreasing sequences of n natural numbers having the property that in each sequence no term divides any other term of the sequence. Let $A = (a_j)$ and $B = (b_j)$ be any two such sequences. We say that A precedes B if for some k, $a_k < b_k$ and $a_i = b_i$ for $i < k$. Find the terms of the first sequence of the set under this ordering.

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6 Suppose that $x_1, x_2, \dots, x_n$ are positive integers for which $x_1 + x_2 + \cdots + x_n = 2(n+1)$. Show that there exists an integer r with $0 \leq r \leq n-1$ for which the following $n-1$ inequalities hold:

$$x_{r+1} + \cdots + x_{r+i} \leq 2i + 1, \qquad \forall i, 1 \leq i \leq n-r;$$

$$x_{r+1} + \cdots + x_n + x_1 + \cdots + x_i \leq 2(n-r+i) + 1, \qquad \forall i, 1 \leq i \leq r-1.$$

Prove that if all the inequalities are strict, then r is unique and that otherwise there are exactly two such r.

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7 Let a be a positive integer and let $\{a_n\}$ be defined by $a_0 = 0$ and

$$a_{n+1} = (a_n + 1)a + (a+1)a_n + 2\sqrt{a(a+1)a_n(a_n+1)} \qquad (n = 1, 2, \dots).$$

Show that for each positive integer n, a_n is a positive integer.

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8 In a test, $3n$ students participate, who are located in three rows of n students in each. The students leave the test room one by one. If $N_1(t), N_2(t), N_3(t)$ denote the numbers of students in the first, second, and third row respectively at time t, find the probability that for each t during the test,

$$|N_i(t) - N_j(t)| < 2, i \neq j, i, j = 1, 2, \dots.$$

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9 Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determine when equality occurs.

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10 Let p and q be integers. Show that there exists an interval I of length $1/q$ and a polynomial P with integral coefficients such that

$$\left|P(x) - \frac{p}{q}\right| < \frac{1}{q^2}$$

for all $x \in I$.

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11 Let $f : [0,1] \to \mathbb{R}$ be continuous and satisfy:

$$\begin{cases} bf(2x) = f(x), & \text{if } 0 \leq x \leq 1/2, \\ f(x) = b + (1-b)f(2x-1), & \text{if } 1/2 \leq x \leq 1, \end{cases}$$

where $b = \frac{1+c}{2+c}$, $c > 0$. Show that $0 < f(x) - x < c$ for every $x, 0 < x < 1$.

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12 Find all functions f defined on the set of positive reals which take positive real values and satisfy: $f(xf(y)) = yf(x)$ for all x, y; and $f(x) \to 0$ as $x \to \infty$.

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13 Let E be the set of 1983^3 points of the space $\mathbb{R}^3$ all three of whose coordinates are integers between 0 and 1982 (including 0 and 1982). A coloring of E is a map from E to the set red, blue. How many colorings of E are there satisfying the following property: The number of red vertices among the 8 vertices of any right-angled parallelepiped is a multiple of 4 ?

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14 Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?

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15 Decide whether there exists a set M of positive integers satisfying the following conditions:

(i) For any natural number $m > 1$ there exist $a, b \in M$ such that $a + b = m$.

(ii) If $a, b, c, d \in M$, $a, b, c, d > 10$ and $a + b = c + d$, then $a = c$ or $a = d$.

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16 Let $F(n)$ be the set of polynomials $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, with $a_0, a_1, ..., a_n \in \mathbb{R}$ and $0 \leq a_0 = a_n \leq a_1 = a_{n-1} \leq \cdots \leq a_{[n/2]} = a_{[(n+1)/2]}$. Prove that if $f \in F(m)$ and $g \in F(n)$, then $fg \in F(m+n)$.

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17 Let $P_1, P_2, \ldots, P_n$ be distinct points of the plane, $n \geq 2$. Prove that

$$\max_{1 \leq i < j \leq n} P_i P_j > \frac{\sqrt{3}}{2}(\sqrt{n} - 1) \min_{1 \leq i < j \leq n} P_i P_j$$

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18 Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where x, y, z are non-negative integers.

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19 Let $(F_n)_{n \geq 1}$ be the Fibonacci sequence $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n (n \geq 1)$, and $P(x)$ the polynomial of degree 990 satisfying

$$P(k) = F_k, \qquad \text{for } k = 992, ..., 1982.$$

Prove that $P(1983) = F_{1983} - 1$.

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20 Find all solutions of the following system of n equations in n variables:

$$x_1|x_1| - (x_1 - a)|x_1 - a| = x_2|x_2|, x_2|x_2| - (x_2 - a)|x_2 - a| = x_3|x_3|, \vdots x_n|x_n| - (x_n - a)|x_n - a| = x_1|x_1|$$

where a is a given number.

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21 Find the greatest integer less than or equal to $\sum_{k=1}^{2^{1983}} k^{\frac{1}{1983} - 1}$.

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22 Let n be a positive integer having at least two different prime factors. Show that there exists a permutation $a_1, a_2, \ldots, a_n$ of the integers $1, 2, \ldots, n$ such that

$$\sum_{k=1}^{n} k \cdot \cos \frac{2\pi a_k}{n} = 0.$$

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23 Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of $P_1 Q_1$ and M_2 the midpoint of $P_2 Q_2$. Prove that $\angle O_1 A O_2 = \angle M_1 A M_2$.

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24 Let d_n be the last nonzero digit of the decimal representation of $n!$. Prove that d_n is aperiodic; that is, there do not exist T and n_0 such that for all $n \geq n_0, d_{n+T} = d_n.$

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25 Prove that every partition of 3-dimensional space into three disjoint subsets has the following property: One of these subsets contains all possible distances; i.e., for every $a \in \mathbb{R}^+$, there are points M and N inside that subset such that distance between M and N is exactly a.
