IMO Shortlist 1984

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1 Find all solutions of the following system of n equations in n variables:

$$x_1|x_1| - (x_1 - a)|x_1 - a| = x_2|x_2|, x_2|x_2| - (x_2 - a)|x_2 - a| = x_3|x_3|, \vdots x_n|x_n| - (x_n - a)|x_n - a| = x_1|x_1|$$

where a is a given number.

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2 Prove:

(a) There are infinitely many triples of positive integers m, n, p such that $4mn - m - n = p^2 - 1$.

(b) There are no positive integers m, n, p such that $4mn - m - n = p^2$.

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3 Find all positive integers n such that

$$n = d_6^2 + d_7^2 - 1,$$

where $1 = d_1 < d_2 < \cdots < d_k = n$ are all positive divisors of the number n.

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4 Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices (where $n > 3$). Let p be its perimeter. Prove that:

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right] \cdot \left[\frac{n+1}{2}\right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding x.

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5 Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, where x, y and z are non-negative real numbers satisfying $x + y + z = 1$.

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6 Let c be a positive integer. The sequence $\{f_n\}$ is defined as follows:

$$f_1 = 1, f_2 = c, f_{n+1} = 2f_n - f_{n-1} + 2 \quad (n \geq 2).$$

Show that for each $k \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that $f_k f_{k+1} = f_r$.

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7 (a) Decide whether the fields of the 8×8 chessboard can be numbered by the numbers $1, 2, \dots, 64$ in such a way that the sum of the four numbers in each of its parts of one of the forms

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is divisible by four.

(b) Solve the analogous problem for

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8 Given points O and A in the plane. Every point in the plane is colored with one of a finite number of colors. Given a point X in the plane, the circle $C(X)$ has center O and radius $OX + \frac{\angle AOX}{OX}$, where $\angle AOX$ is measured in radians in the range $[0, 2\pi)$. Prove that we can find a point X, not on OA, such that its color appears on the circumference of the circle $C(X)$.

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9 Let a, b, c be positive numbers with $\sqrt{a} + \sqrt{b} + \sqrt{c} = \frac{\sqrt{3}}{2}$. Prove that the system of equations

$$\sqrt{y-a} + \sqrt{z-a} = 1,$$

$$\sqrt{z-b} + \sqrt{x-b} = 1,$$

$$\sqrt{x-c} + \sqrt{y-c} = 1$$

has exactly one solution (x, y, z) in real numbers.

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10 Prove that the product of five consecutive positive integers cannot be the square of an integer.

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11 Let n be a positive integer and $a_1, a_2, \dots, a_{2n}$ mutually distinct integers. Find all integers x satisfying

$$(x - a_1) \cdot (x - a_2) \cdots (x - a_{2n}) = (-1)^n (n!)^2.$$

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12 Find one pair of positive integers a, b such that $ab(a+b)$ is not divisible by 7, but $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

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13 Prove that the volume of a tetrahedron inscribed in a right circular cylinder of volume 1 does not exceed $\frac{2}{3\pi}$.

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14 Let $ABCD$ be a convex quadrilateral with the line CD being tangent to the circle on diameter AB. Prove that the line AB is tangent to the circle on diameter CD if and only if the lines BC and AD are parallel.

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15 Angles of a given triangle ABC are all smaller than $120°$. Equilateral triangles AFB, BDC and CEA are constructed in the exterior of ABC.

(a) Prove that the lines AD, BE, and CF pass through one point S.

(b) Prove that $SD + SE + SF = 2(SA + SB + SC)$.

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16 Let a, b, c, d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.

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17 In a permutation $(x_1, x_2, \dots, x_n)$ of the set $1, 2, \dots, n$ we call a pair (x_i, x_j) discordant if $i < j$ and $x_i > x_j$. Let $d(n, k)$ be the number of such permutations with exactly k discordant pairs. Find $d(n, 2)$ and $d(n, 3)$.

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18 Inside triangle ABC there are three circles k_1, k_2, k_3 each of which is tangent to two sides of the triangle and to its incircle k. The radii of k_1, k_2, k_3 are $1, 4$, and 9. Determine the radius of k.

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19 The harmonic table is a triangular array:

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$\frac{1}{2} \quad \frac{1}{2}$

$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3}$

$\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4}$

Where $a_{n,1} = \frac{1}{n}$ and $a_{n,k+1} = a_{n-1,k} - a_{n,k}$ for $1 \leq k \leq n-1$. Find the harmonic mean of the 1985^{th} row.

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20 Determine all pairs (a, b) of positive real numbers with $a \neq 1$ such that

$$\log_a b < \log_{a+1}(b + 1).$$
