IMO Shortlist 1985

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1 Given a set M of 1985 positive integers, none of which has a prime divisor larger than 26, prove that the set has four distinct elements whose geometric mean is an integer.

2 A polyhedron has 12 faces and is such that:

(i) all faces are isosceles triangles,
(ii) all edges have length either x or y,
(iii) at each vertex either 3 or 6 edges meet, and
(iv) all dihedral angles are equal.

Find the ratio x/y.

3 For any polynomial $P(x) = a_0 + a_1x + \ldots + a_kx^k$ with integer coefficients, the number of odd coefficients is denoted by $o(P)$. For $i - 0, 1, 2, \ldots$ let $Q_i(x) = (1 + x)^i$. Prove that if $i_1, i_2, \ldots, i_n$ are integers satisfying $0 \leq i_1 < i_2 < \ldots < i_n$, then:

$$o(Q_{i_1} + Q_{i_2} + \ldots + Q_{i_n}) \geq o(Q_{i_1}).$$

4 Each of the numbers in the set $N = \{1, 2, 3, \cdots, n-1\}$, where $n \geq 3$, is colored with one of two colors, say red or black, so that:

(i) i and $n - i$ always receive the same color, and

(ii) for some $j \in N$, relatively prime to n, i and $|j - i|$ receive the same color for all $i \in N, i \neq j$.

Prove that all numbers in N must receive the same color.

5 Let D be the interior of the circle C and let $A \in C$. Show that the function $f : D \to \mathbb{R}, f(M) = \frac{|MA|}{|MM'|}$ where $M' = AM \cap C$, is strictly convex; i.e., $f(P) < \frac{f(M_1)+f(M_2)}{2}, \forall M_1, M_2 \in D, M_1 \neq M_2$ where P is the midpoint of the segment M_1M_2.

6 Let $x_n = \sqrt[2]{2 + \sqrt[3]{3 + \cdots + \sqrt[n]{n}}}$. Prove that

$$x_{n+1} - x_n < \frac{1}{n!} \quad n = 2, 3, \cdots$$

7   The positive integers $x_1, \cdots, x_n, n \geq 3$, satisfy $x_1 < x_2 < \cdots < x_n < 2x_1$. Set $P = x_1 x_2 \cdots x_n$.
Prove that if p is a prime number, k a positive integer, and P is divisible by pk, then $\frac{P}{p^k} \geq n!$.

8   Let A be a set of n points in the space. From the family of all segments with endpoints in
A, q segments have been selected and colored yellow. Suppose that all yellow segments are
of different length. Prove that there exists a polygonal line composed of m yellow segments,
where $m \geq \frac{2q}{n}$, arranged in order of increasing length.

9   Determine the radius of a sphere S that passes through the centroids of each face of a given
tetrahedron T inscribed in a unit sphere with center O. Also, determine the distance from O to
the center of S as a function of the edges of T.

10   Prove that for every point M on the surface of a regular tetrahedron there exists a point M'
such that there are at least three different curves on the surface joining M to M' with the
smallest possible length among all curves on the surface joining M to M'.

11   Find a method by which one can compute the coefficients of $P(x) = x^6 + a_1 x^5 + \cdots + a_6$ from
the roots of $P(x) = 0$ by performing not more than 15 additions and 15 multiplications.

12   A sequence of polynomials $P_m(x, y, z), m = 0, 1, 2, \cdots$, in x, y, and z is defined by $P_0(x, y, z) = 1$
and by
$$P_m(x, y, z) = (x + z)(y + z)P_{m-1}(x, y, z + 1) - z^2 P_{m-1}(x, y, z)$$
for $m > 0$. Prove that each $P_m(x, y, z)$ is symmetric, in other words, is unaltered by any permu-
tation of x, y, z.

13   Let m boxes be given, with some balls in each box. Let $n < m$ be a given integer. The following
operation is performed: choose n of the boxes and put 1 ball in each of them. Prove:

(a) If m and n are relatively prime, then it is possible, by performing the operation a finite number
of times, to arrive at the situation that all the boxes contain an equal number of balls.

(b) If m and n are not relatively prime, there exist initial distributions of balls in the boxes such
that an equal distribution is not possible to achieve.

14   A set of 1985 points is distributed around the circumference of a circle and each of the points
is marked with 1 or -1. A point is called good if the partial sums that can be formed by starting
at that point and proceeding around the circle for any distance in either direction are all strictly
positive. Show that if the number of points marked with -1 is less than 662, there must be at
least one good point.

15   Let K and K' be two squares in the same plane, their sides of equal length. Is it possible to
decompose K into a finite number of triangles $T_1, T_2, \ldots, T_p$ with mutually disjoint interiors

and find translations $t_1, t_2, \ldots, t_p$ such that

$$K' = \bigcup_{i=1}^{p} t_i(T_i) \ ?$$

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16 If possible, construct an equilateral triangle whose three vertices are on three given circles.

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17 The sequence $f_1, f_2, \cdots, f_n, \cdots$ of functions is defined for $x > 0$ recursively by

$$f_1(x) = x, \quad f_{n+1}(x) = f_n(x)\left(f_n(x) + \frac{1}{n}\right)$$

Prove that there exists one and only one positive number a such that $0 < f_n(a) < f_{n+1}(a) < 1$ for all integers $n \geq 1$.

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18 Let $x_1, x_2, \cdots, x_n$ be positive numbers. Prove that

$$\frac{x_1^2}{x_1^2 + x_2 x_3} + \frac{x_2^2}{x_2^2 + x_3 x_4} + \cdots + \frac{x_{n-1}^2}{x_{n-1}^2 + x_n x_1} + \frac{x_n^2}{x_n^2 + x_1 x_2} \leq n - 1$$

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19 For which integers $n \geq 3$ does there exist a regular n-gon in the plane such that all its vertices have integer coordinates in a rectangular coordinate system?

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20 A circle whose center is on the side ED of the cyclic quadrilateral $BCDE$ touches the other three sides. Prove that $EB + CD = ED$.

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21 The tangents at B and C to the circumcircle of the acute-angled triangle ABC meet at X. Let M be the midpoint of BC. Prove that

(a) $\angle BAM = \angle CAX$, and

(b) $\frac{AM}{AX} = \cos \angle BAC$.

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22 A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.