IMO Shortlist 1986

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by orl, Amir Hossein

1 Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) with center O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, with X remaining inside the polygon. Find the locus of X.

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2 Let $f(x) = x^n$ where n is a fixed positive integer and $x = 1, 2, \cdots$. Is the decimal expansion $a = 0.f(1)f(2)f(3)...$ rational for any value of n ?

The decimal expansion of a is defined as follows: If $f(x) = d_1(x)d_2(x)\cdots d_{r(x)}(x)$ is the decimal expansion of $f(x)$, then $a = 0.1d_1(2)d_2(2)\cdots d_{r(2)}(2)d_1(3)...d_{r(3)}(3)d_1(4)\cdots$.

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3 Let A, B, and C be three points on the edge of a circular chord such that B is due west of C and ABC is an equilateral triangle whose side is 86 meters long. A boy swam from A directly toward B. After covering a distance of x meters, he turned and swam westward, reaching the shore after covering a distance of y meters. If x and y are both positive integers, determine y.

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4 Provided the equation $xyz = p^n(x + y + z)$ where $p \geq 3$ is a prime and $n \in \mathbb{N}$. Prove that the equation has at least $3n + 3$ different solutions (x, y, z) with natural numbers x, y, z and $x < y < z$. Prove the same for $p > 3$ being an odd integer.

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5 Let d be any positive integer not equal to $2, 5$ or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

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6 Find four positive integers each not exceeding 70000 and each having more than 100 divisors.

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7 Let real numbers $x_1, x_2, \cdots , x_n$ satisfy $0 < x_1 < x_2 < \cdots < x_n < 1$ and set $x_0 = 0, x_{n+1} = 1$. Suppose that these numbers satisfy the following system of equations:

$$\sum_{j=0, j\neq i}^{n+1} \frac{1}{x_i - x_j} = 0 \quad \text{where } i = 1, 2, ..., n.$$

Prove that $x_{n+1-i} = 1 - x_i$ for $i = 1, 2, ..., n$.

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8 From a collection of n persons q distinct two-member teams are selected and ranked $1, \cdots , q$ (no ties). Let m be the least integer larger than or equal to $2q/n$. Show that there are m distinct teams that may be listed so that :
(i) each pair of consecutive teams on the list have one member in common and
(ii) the chain of teams on the list are in rank order.

Alternative formulation.
Given a graph with n vertices and q edges numbered $1, \cdots, q$, show that there exists a chain of m edges, $m \geq \frac{2q}{n}$, each two consecutive edges having a common vertex, arranged monotonically with respect to the numbering.

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9 Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line L parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on L is not greater than 1?

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10 Three persons A, B, C, are playing the following game:

A k-element subset of the set $\{1, ..., 1986\}$ is randomly chosen, with an equal probability of each choice, where k is a fixed positive integer less than or equal to 1986. The winner is A, B or C, respectively, if the sum of the chosen numbers leaves a remainder of $0, 1$, or 2 when divided by 3.

For what values of k is this game a fair one? (A game is fair if the three outcomes are equally probable.)

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11 Let $f(n)$ be the least number of distinct points in the plane such that for each $k = 1, 2, \cdots, n$ there exists a straight line containing exactly k of these points. Find an explicit expression for $f(n)$.

Simplified version.

Show that $f(n) = \left[\frac{n+1}{2}\right] \left[\frac{n+2}{2}\right]$. Where $[x]$ denoting the greatest integer not exceeding x.

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12 To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

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13 A particle moves from $(0, 0)$ to (n, n) directed by a fair coin. For each head it moves one step east and for each tail it moves one step north. At $(n, y), y < n$, it stays there if a head comes up and at $(x, n), x < n$, it stays there if a tail comes up. Let$k$ be a fixed positive integer. Find the probability that the particle needs exactly $2n + k$ tosses to reach (n, n).

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14 The circle inscribed in a triangle ABC touches the sides BC, CA, AB in D, E, F, respectively, and X, Y, Z are the midpoints of EF, FD, DE, respectively. Prove that the centers of the inscribed circle and of the circles around XYZ and ABC are collinear.

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15 Let $ABCD$ be a convex quadrilateral whose vertices do not lie on a circle. Let $A'B'C'D'$ be a quadrangle such that A', B', C', D' are the centers of the circumcircles of triangles BCD, ACD, ABD, and ABC. We write $T(ABCD) = A'B'C'D'$. Let us define $A''B''C''D'' = T(A'B'C'D') = T(T(ABCD))$.

(a) Prove that $ABCD$ and $A''B''C''D''$ are similar.

(b) The ratio of similitude depends on the size of the angles of $ABCD$. Determine this ratio.

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16 Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) with center O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, with X remaining inside the polygon. Find the locus of X.

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17 Given a point P_0 in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all $s \geq 4$. Construct a set of points $P_1, P_2, P_3, \ldots$ such that P_{k+1} is the image of P_k under a rotation center A_{k+1} through an angle 120^o clockwise for $k = 0, 1, 2, \ldots$. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

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18 Let AX, BY, CZ be three cevians concurrent at an interior point D of a triangle ABC. Prove that if two of the quadrangles $DYAZ, DZBX, DXCY$ are circumscribable, so is the third.

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19 A tetrahedron $ABCD$ is given such that $AD = BC = a; AC = BD = b; AB \cdot CD = c^2$. Let $f(P) = AP + BP + CP + DP$, where P is an arbitrary point in space. Compute the least value of $f(P)$.

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20 Prove that the sum of the face angles at each vertex of a tetrahedron is a straight angle if and only if the faces are congruent triangles.

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21 Let $ABCD$ be a tetrahedron having each sum of opposite sides equal to 1. Prove that

$$r_A + r_B + r_C + r_D \leq \frac{\sqrt{3}}{3}$$

where r_A, r_B, r_C, r_D are the inradii of the faces, equality holding only if $ABCD$ is regular.
