IMO Shortlist 1987

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by Amir Hossein

1 Let f be a function that satisfies the following conditions:

(i) If $x > y$ and $f(y) - y \geq v \geq f(x) - x$, then $f(z) = v + z$, for some number z between x and y. (ii) The equation $f(x) = 0$ has at least one solution, and among the solutions of this equation, there is one that is not smaller than all the other solutions; (iii) $f(0) = 1$. (iv) $f(1987) \leq 1988$. (v) $f(x)f(y) = f(xf(y) + yf(x) - xy)$.

Find $f(1987)$.

Proposed by Australia.

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2 At a party attended by n married couples, each person talks to everyone else at the party except his or her spouse. The conversations involve sets of persons or cliques $C_1, C_2, \cdots, C_k$ with the following property: no couple are members of the same clique, but for every other pair of persons there is exactly one clique to which both members belong. Prove that if $n \geq 4$, then $k \geq 2n$.

Proposed by USA.

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3 Does there exist a second-degree polynomial $p(x, y)$ in two variables such that every non-negative integer n equals $p(k, m)$ for one and only one ordered pair (k, m) of non-negative integers?

Proposed by Finland.

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4 Let $ABCDEFGH$ be a parallelepiped with $AE \parallel BF \parallel CG \parallel DH$. Prove the inequality

$$AF + AH + AC \leq AB + AD + AE + AG.$$

In what cases does equality hold?

Proposed by France.

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5 Find, with proof, the point P in the interior of an acute-angled triangle ABC for which $BL^2 + CM^2 + AN^2$ is a minimum, where L, M, N are the feet of the perpendiculars from P to BC, CA, AB respectively.

Proposed by United Kingdom.

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6 Show that if a, b, c are the lengths of the sides of a triangle and if $2S = a + b + c$, then

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \left(\frac{2}{3}\right)^{n-2} S^{n-1} \quad \forall n \in \mathbb{N}$$

Proposed by Greece.

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7 Given five real numbers u_0, u_1, u_2, u_3, u_4, prove that it is always possible to find five real numbers $v0, v_1, v_2, v_3, v_4$ that satisfy the following conditions:

(i) $u_i - v_i \in \mathbb{N}, \quad 0 \leq i \leq 4$

(ii) $\sum_{0 \leq i < j \leq 4} (v_i - v_j)^2 < 4.$

Proposed by Netherlands.

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8 (a) Let $\gcd(m, k) = 1$. Prove that there exist integers $a_1, a_2, ..., a_m$ and $b_1, b_2, ..., b_k$ such that each product $a_i b_j$ ($i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, k$) gives a different residue when divided by $mk.$

(b) Let $\gcd(m, k) > 1$. Prove that for any integers $a_1, a_2, ..., a_m$ and $b_1, b_2, ..., b_k$ there must be two products $a_i b_j$ and $a_s b_t$ ($(i, j) \neq (s, t)$) that give the same residue when divided by mk.

Proposed by Hungary.

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9 Does there exist a set M in usual Euclidean space such that for every plane λ the intersection $M \cap \lambda$ is finite and nonempty ?

Proposed by Hungary.

I'm not sure I'm posting this in a right Forum.

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10 Let S_1 and S_2 be two spheres with distinct radii that touch externally. The spheres lie inside a cone C, and each sphere touches the cone in a full circle. Inside the cone there are n additional solid spheres arranged in a ring in such a way that each solid sphere touches the cone C, both of the spheres S_1 and S_2 externally, as well as the two neighboring solid spheres. What are the possible values of n?

Proposed by Iceland.

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11 Find the number of partitions of the set $\{1, 2, \cdots, n\}$ into three subsets A_1, A_2, A_3, some of which may be empty, such that the following conditions are satisfied:

(i) After the elements of every subset have been put in ascending order, every two consecutive elements of any subset have different parity.

(ii) If A_1, A_2, A_3 are all nonempty, then in exactly one of them the minimal number is even .

Proposed by Poland.

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12 Given a nonequilateral triangle ABC, the vertices listed counterclockwise, find the locus of the centroids of the equilateral triangles $A'B'C'$ (the vertices listed counterclockwise) for which the triples of points A, B', C'; A', B, C'; and A', B', C are collinear.

Proposed by Poland.

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13 Is it possible to put 1987 points in the Euclidean plane such that the distance between each pair of points is irrational and each three points determine a non-degenerate triangle with rational area? *(IMO Problem 5)*

Proposed by Germany, DR

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14 How many words with n digits can be formed from the alphabet $\{0, 1, 2, 3, 4\}$, if neighboring digits must differ by exactly one?

Proposed by Germany, FR.

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15 Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $|a_i| \leq k - 1$ for all i, and $|a_1 x_1 + a_2 x_2 + \ldots + a_n x_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}$. *(IMO Problem 3)*

Proposed by Germany, FR

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16 Let $p_n(k)$ be the number of permutations of the set $\{1, 2, 3, \ldots, n\}$ which have exactly k fixed points. Prove that $\sum_{k=0}^{n} k p_n(k) = n!$.*(IMO Problem 1)*

Original formulation

Let S be a set of n elements. We denote the number of all permutations of S that have exactly k fixed points by $p_n(k)$. Prove:

(a) $\sum_{k=0}^{n} k p_n(k) = n!$;

(b) $\sum_{k=0}^{n} (k-1)^2 p_n(k) = n!$

Proposed by Germany, FR

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17 Prove that there exists a four-coloring of the set $M = \{1, 2, \cdots, 1987\}$ such that any arithmetic progression with 10 terms in the set M is not monochromatic.

Alternative formulation

Let $M = \{1, 2, \cdots, 1987\}$. Prove that there is a function $f : M \to \{1, 2, 3, 4\}$ that is not constant on every set of 10 terms from M that form an arithmetic progression.

Proposed by Romania

18 For any integer $r \geq 1$, determine the smallest integer $h(r) \geq 1$ such that for any partition of the set $\{1, 2, \cdots, h(r)\}$ into r classes, there are integers $a \geq 0 \,; 1 \leq x \leq y$, such that $a + x, a + y, a + x + y$ belong to the same class.

Proposed by Romania

19 Let α, β, γ be positive real numbers such that $\alpha + \beta + \gamma < \pi$, $\alpha + \beta > \gamma$,$\beta + \gamma > \alpha$, $\gamma + \alpha > \beta$. Prove that with the segments of lengths $\sin\alpha, \sin\beta, \sin\gamma$ we can construct a triangle and that its area is not greater than

$$A = \frac{1}{8}\left(\sin 2\alpha + \sin 2\beta + \sin 2\gamma\right).$$

Proposed by Soviet Union

20 Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.*(IMO Problem 6)*

Original Formulation

Let $f(x) = x^2 + x + p, p \in \mathbb{N}$. Prove that if the numbers $f(0), f(1), \cdots, f(\sqrt{\frac{p}{3}})$ are primes, then all the numbers $f(0), f(1), \cdots, f(p-2)$ are primes.

Proposed by Soviet Union.

21 In an acute-angled triangle ABC the interior bisector of angle A meets BC at L and meets the circumcircle of ABC again at N. From L perpendiculars are drawn to AB and AC, with feet K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas.*(IMO Problem 2)*

Proposed by Soviet Union.

22 Does there exist a function $f : \mathbb{N} \to \mathbb{N}$, such that $f(f(n)) = n + 1987$ for every natural number n? *(IMO Problem 4)*

Proposed by Vietnam.

23 Prove that for every natural number k ($k \geq 2$) there exists an irrational number r such that for every natural number m,

$$[r^m] \equiv -1 \pmod{k}.$$

Remark. An easier variant: Find r as a root of a polynomial of second degree with integer coefficients.

Proposed by Yugoslavia.