IMO Shortlist 1988

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1   An integer sequence is defined by

$$a_n = 2a_{n-1} + a_{n-2}, \quad (n > 1), \quad a_0 = 0, a_1 = 1.$$

Prove that 2^k divides a_n if and only if 2^k divides n.

2   Let n be a positive integer. Find the number of odd coefficients of the polynomial

$$u_n(x) = (x^2 + x + 1)^n.$$

3   The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B and C meet the circle again at A', B' and C' respectively. Prove that the area of triangle $A'B'C'$ is greater than or equal to the area of triangle ABC.

4   An $n \times n, n \geq 2$ chessboard is numbered by the numbers $1, 2, \ldots, n^2$ (and every number occurs). Prove that there exist two neighbouring (with common edge) squares such that their numbers differ by at least n.

5   Let n be an even positive integer. Let $A_1, A_2, \ldots, A_{n+1}$ be sets having n elements each such that any two of them have exactly one element in common while every element of their union belongs to at least two of the given sets. For which n can one assign to every element of the union one of the numbers 0 and 1 in such a manner that each of the sets has exactly $\frac{n}{2}$ zeros?

6   In a given tedrahedron $ABCD$ let K and L be the centres of edges AB and CD respectively. Prove that every plane that contains the line KL divides the tedrahedron into two parts of equal volume.

7   Let a be the greatest positive root of the equation $x^3 - 3 \cdot x^2 + 1 = 0$. Show that $\left[a^{1788}\right]$ and $\left[a^{1988}\right]$ are both divisible by 17. Here $[x]$ denotes the integer part of x.

8   Let $u_1, u_2, \ldots, u_m$ be m vectors in the plane, each of length ≤ 1, with zero sum. Show that one can arrange $u_1, u_2, \ldots, u_m$ as a sequence $v_1, v_2, \ldots, v_m$ such that each partial sum $v_1, v_1 + v_2, v_1 + v_2 + v_3, \ldots, v_1, v_2, \ldots, v_m$ has length less than or equal to $\sqrt{5}$.

9   Let a and b be two positive integers such that $a \cdot b + 1$ divides $a^2 + b^2$. Show that $\frac{a^2+b^2}{a \cdot b+1}$ is a perfect square.

10   Let $N = \{1, 2 \ldots, n\}, n \geq 2$. A collection $F = \{A_1, \ldots, A_t\}$ of subsets $A_i \subseteq N$, $i = 1, \ldots, t$, is said to be separating, if for every pair $\{x, y\} \subseteq N$, there is a set $A_i \in F$ so that $A_i \cap \{x, y\}$ contains just one element. F is said to be covering, if every element of N is contained in at least one set $A_i \in F$. What is the smallest value $f(n)$ of t, so there is a set $F = \{A_1, \ldots, A_t\}$ which is simultaneously separating and covering?

11   The lock of a safe consists of 3 wheels, each of which may be set in 8 different ways positions. Due to a defect in the safe mechanism the door will open if any two of the three wheels are in the correct position. What is the smallest number of combinations which must be tried if one is to guarantee being able to open the safe (assuming the "right combination" is not known)?

12   In a triangle ABC, choose any points $K \in BC, L \in AC, M \in AB, N \in LM, R \in MK$ and $F \in KL$. If $E_1, E_2, E_3, E_4, E_5, E_6$ and E denote the areas of the triangles $AMR, CKR, BKF, ALF, BNM, CLN$ and ABC respectively, show that

$$E \geq 8 \cdot \sqrt[6]{E_1 E_2 E_3 E_4 E_5 E_6}.$$

13   In a right-angled triangle ABC let AD be the altitude drawn to the hypotenuse and let the straight line joining the incentres of the triangles ABD, ACD intersect the sides AB, AC at the points K, L respectively. If E and E_1 dnote the areas of triangles ABC and AKL respectively, show that

$$\frac{E}{E_1} \geq 2.$$

14   For what values of n does there exist an $n \times n$ array of entries -1, 0 or 1 such that the $2 \cdot n$ sums obtained by summing the elements of the rows and the columns are all different?

15   Let ABC be an acute-angled triangle. The lines L_A, L_B and L_C are constructed through the vertices A, B and C respectively according the following prescription: Let H be the foot of the altitude drawn from the vertex A to the side BC; let S_A be the circle with diameter AH; let S_A meet the sides AB and AC at M and N respectively, where M and N are distinct from A; then let L_A be the line through A perpendicular to MN. The lines L_B and L_C are constructed similarly. Prove that the lines L_A, L_B and L_C are concurrent.

16   Show that the solution set of the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose length is 1988.

17   In the convex pentagon $ABCDE$, the sides BC, CD, DE are equal. Moreover each diagonal of the pentagon is parallel to a side (AC is parallel to DE, BD is parallel to AE etc.). Prove that $ABCDE$ is a regular pentagon.

18   Consider 2 concentric circle radii R and r $(R > r)$ with centre O. Fix P on the small circle and consider the variable chord PA of the small circle. Points B and C lie on the large circle; B, P, C are collinear and BC is perpendicular to AP.

i.) For which values of $\angle OPA$ is the sum $BC^2 + CA^2 + AB^2$ extremal?

ii.) What are the possible positions of the midpoints U of BA and V of AC as $\angle OPA$ varies?

19   Let $f(n)$ be a function defined on the set of all positive integers and having its values in the same set. Suppose that $f(f(n)+f(m)) = m+n$ for all positive integers n, m. Find the possible value for $f(1988)$.

20   Find the least natural number n such that, if the set $\{1, 2, \ldots, n\}$ is arbitrarily divided into two non-intersecting subsets, then one of the subsets contains 3 distinct numbers such that the product of two of them equals the third.

21   Forty-nine students solve a set of 3 problems. The score for each problem is a whole number of points from 0 to 7. Prove that there exist two students A and B such that, for each problem, A will score at least as many points as B.

22   Let p be the product of two consecutive integers greater than 2. Show that there are no integers $x_1, x_2, \ldots, x_p$ satisfying the equation

$$\sum_{i=1}^{p} x_i^2 - \frac{4}{4 \cdot p + 1} \left(\sum_{i=1}^{p} x_i \right)^2 = 1$$

OR

Show that there are only two values of p for which there are integers $x_1, x_2, \ldots, x_p$ satisfying

$$\sum_{i=1}^{p} x_i^2 - \frac{4}{4 \cdot p + 1} \left(\sum_{i=1}^{p} x_i \right)^2 = 1$$

23   Let Q be the centre of the inscribed circle of a triangle ABC. Prove that for any point P,

$$a(PA)^2 + b(PB)^2 + c(PC)^2 = a(QA)^2 + b(QB)^2 + c(QC)^2 + (a+b+c)(QP)^2,$$

where $a = BC, b = CA$ and $c = AB$.

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24   Let $\{a_k\}_1^\infty$ be a sequence of non-negative real numbers such that:

$$a_k - 2a_{k+1} + a_{k+2} \geq 0$$

and $\sum_{j=1}^{k} a_j \leq 1$ for all $k = 1, 2, \ldots$. Prove that:

$$0 \leq a_k - a_{k+1} < \frac{2}{k^2}$$

for all $k = 1, 2, \ldots$.

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25   A positive integer is called a **double number** if its decimal representation consists of a block of digits, not commencing with 0, followed immediately by an identical block. So, for instance, 360360 is a double number, but 36036 is not. Show that there are infinitely many double numbers which are perfect squares.

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26   A function f defined on the positive integers (and taking positive integers values) is given by:
$$f(1) = 1, f(3) = 3$$
$$f(2 \cdot n) = f(n)$$
$$f(4 \cdot n + 1) = 2 \cdot f(2 \cdot n + 1) - f(n)$$
$$f(4 \cdot n + 3) = 3 \cdot f(2 \cdot n + 1) - 2 \cdot f(n),$$

for all positive integers n. Determine with proof the number of positive integers ≤ 1988 for which $f(n) = n$.

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27   Let ABC be an acute-angled triangle. Let L be any line in the plane of the triangle ABC. Denote by u, v, w the lengths of the perpendiculars to L from A, B, C respectively. Prove the inequality $u^2 \cdot \tan A + v^2 \cdot \tan B + w^2 \cdot \tan C \geq 2 \cdot S$, where S is the area of the triangle ABC. Determine the lines L for which equality holds.

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28   The sequence $\{a_n\}$ of integers is defined by

$$a_1 = 2, a_2 = 7$$

and

$$-\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leq \ , n \geq 2.$$

Prove that a_n is odd for all $n > 1$.

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29   A number of signal lights are equally spaced along a one-way railroad track, labeled in oder $1, 2, \ldots, N, N \geq 2$. As a safety rule, a train is not allowed to pass a signal if any other train is in motion on the length of track between it and the following signal. However, there is no limit to the number of trains that can be parked motionless at a signal, one behind the other. (Assume

the trains have zero length.) A series of K freight trains must be driven from Signal 1 to Signal N. Each train travels at a distinct but constant spped at all times when it is not blocked by the safety rule. Show that, regardless of the order in which the trains are arranged, the same time will elapse between the first train's departure from Signal 1 and the last train's arrival at Signal N.

30   A point M is chosen on the side AC of the triangle ABC in such a way that the radii of the circles inscribed in the triangles ABM and BMC are equal. Prove that

$$BM^2 = X \cot \left(\frac{B}{2} \right)$$

where X is the area of triangle ABC.

31   Around a circular table an even number of persons have a discussion. After a break they sit again around the circular table in a different order. Prove that there are at least two people such that the number of participants sitting between them before and after a break is the same.