IMO Shortlist 1989

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1 ABC is a triangle, the bisector of angle A meets the circumcircle of triangle ABC in A_1, points B_1 and C_1 are defined similarly. Let AA_1 meet the lines that bisect the two external angles at B and C in A_0. Define B_0 and C_0 similarly. Prove that the area of triangle $A_0B_0C_0 = 2\cdot$ area of hexagon $AC_1BA_1CB_1 \geq 4\cdot$ area of triangle ABC.

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2 Ali Barber, the carpet merchant, has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. If the two rooms have dimensions of 38 feet by 55 feet and 50 feet by 55 feet, what are the carpet dimensions?

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3 Ali Barber, the carpet merchant, has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. He knows that the sides of the carpet are integral numbers of feet and that his two storerooms have the same (unknown) length, but widths of 38 feet and 50 feet respectively. What are the carpet dimensions?

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4 Prove that $\forall n > 1, n \in \mathbb{N}$ the equation

$$\sum_{k=1}^{n} \frac{x^k}{k!} + 1 = 0$$

has no rational roots.

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5 Find the roots $r_i \in \mathbb{R}$ of the polynomial

$$p(x) = x^n + n \cdot x^{n-1} + a_2 \cdot x^{n-2} + \ldots + a_n$$

satisfying

$$\sum_{k=1}^{16} r_k^{16} = n.$$

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6 For a triangle $ABC,$ let k be its circumcircle with radius $r.$ The bisectors of the inner angles $A, B,$ and C of the triangle intersect respectively the circle k again at points $A', B',$ and $C'.$ Prove the inequality

$$16Q^3 \geq 27r^4P,$$

where Q and P are the areas of the triangles $A'B'C'$ and ABC respectively.

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7 Show that any two points lying inside a regular $n-$gon E can be joined by two circular arcs lying inside E and meeting at an angle of at least $\left(1 - \frac{2}{n}\right) \cdot \pi.$

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8 Let R be a rectangle that is the union of a finite number of rectangles $R_i, 1 \leq i \leq n,$ satisfying the following conditions:

(i) The sides of every rectangle R_i are parallel to the sides of $R.$
(ii) The interiors of any two different rectangles R_i are disjoint.
(iii) Each rectangle R_i has at least one side of integral length.

Prove that R has at least one side of integral length.

Variant: Same problem but with rectangular parallelepipeds having at least one integral side.

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9 $\forall n > 0, n \in \mathbb{Z},$ there exists uniquely determined integers $a_n, b_n, c_n \in \mathbb{Z}$ such

$$\left(1 + 4 \cdot \sqrt[3]{2} - 4 \cdot \sqrt[3]{4}\right)^n = a_n + b_n \cdot \sqrt[3]{2} + c_n \cdot \sqrt[3]{4}.$$

Prove that $c_n = 0$ implies $n = 0.$

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10 Let $g : \mathbb{C} \rightarrow \mathbb{C}$, $\omega \in \mathbb{C}$, $a \in \mathbb{C}$, $\omega^3 = 1$, and $\omega \neq 1$. Show that there is one and only one function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$f(z) + f(\omega z + a) = g(z), z \in \mathbb{C}$$

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11 Define sequence (a_n) by $\sum_{d|n} a_d = 2^n$. Show that $n|a_n$.

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12 There are n cars waiting at distinct points of a circular race track. At the starting signal each car starts. Each car may choose arbitrarily which of the two possible directions to go. Each car has the same constant speed. Whenever two cars meet they both change direction (but not speed). Show that at some time each car is back at its starting point.

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13 Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC.$ There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC.$ Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$

14 A bicentric quadrilateral is one that is both inscribable in and circumscribable about a circle, i.e. both the incircle and circumcircle exists. Show that for such a quadrilateral, the centers of the two associated circles are collinear with the point of intersection of the diagonals.

15 Let $a, b, c, d, m, n \in \mathbb{Z}^+$ such that

$$a^2 + b^2 + c^2 + d^2 = 1989,$$

$$a + b + c + d = m^2,$$

and the largest of a, b, c, d is n^2. Determine, with proof, the values of m and n.

16 The set $\{a_0, a_1, \ldots, a_n\}$ of real numbers satisfies the following conditions:

(i) $a_0 = a_n = 0,$
(ii) for $1 \leq k \leq n - 1,$

$$a_k = c + \sum_{i=k}^{n-1} a_{i-k} \cdot (a_i + a_{i+1})$$

Prove that $c \leq \frac{1}{4n}$.

17 Given seven points in the plane, some of them are connected by segments such that:

(i) among any three of the given points, two are connected by a segment;
(ii) the number of segments is minimal.

How many segments does a figure satisfying **(i)** and **(ii)** have? Give an example of such a figure.

18 Given a convex polygon $A_1A_2\ldots A_n$ with area S and a point M in the same plane, determine the area of polygon $M_1M_2\ldots M_n$, where M_i is the image of M under rotation $R^{\alpha}_{A_i}$ around A_i by $\alpha_i, i = 1, 2, \ldots, n$.

19 A natural number is written in each square of an $m \times n$ chess board. The allowed move is to add an integer k to each of two adjacent numbers in such a way that non-negative numbers are obtained. (Two squares are adjacent if they have a common side.) Find a necessary and sufficient condition for it to be possible for all the numbers to be zero after finitely many operations.

20 Let n and k be positive integers and let S be a set of n points in the plane such that

i.) no three points of S are collinear, and

ii.) for every point P of S there are at least k points of S equidistant from P.

Prove that:

$$k < \frac{1}{2} + \sqrt{2 \cdot n}$$

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21 Prove that the intersection of a plane and a regular tetrahedron can be an obtuse-angled triangle and that the obtuse angle in any such triangle is always smaller than 120°.

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22 Prove that in the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets $A_i, \{i = 1, 2, \ldots, 117\}$ such that

i.) each A_i contains 17 elements

ii.) the sum of all the elements in each A_i is the same.

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23 A permutation $\{x_1, x_2, \ldots, x_{2n}\}$ of the set $\{1, 2, \ldots, 2n\}$ where n is a positive integer, is said to have property T if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that, for each n, there are more permutations with property T than without.

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24 For points $A_1, \ldots, A_5$ on the sphere of radius 1, what is the maximum value that $min_{1 \leq i,j \leq 5} A_iA_j$ can take? Determine all configurations for which this maximum is attained. (Or: determine the diameter of any set $\{A_1, \ldots, A_5\}$ for which this maximum is attained.)

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25 Let $a, b \in \mathbb{Z}$ which are not perfect squares. Prove that if

$$x^2 - ay^2 - bz^2 + abw^2 = 0$$

has a nontrivial solution in integers, then so does

$$x^2 - ay^2 - bz^2 = 0.$$

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26 Let $n \in \mathbb{Z}^+$ and let $a, b \in \mathbb{R}$. Determine the range of x_0 for which

$$\sum_{i=0}^{n} x_i = a \text{ and } \sum_{i=0}^{n} x_i^2 = b,$$

where $x_0, x_1, \ldots, x_n$ are real variables.

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27 Let m be a positive odd integer, $m > 2$. Find the smallest positive integer n such that 2^{1989} divides $m^n - 1$.

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28 Consider in a plane P the points O, A_1, A_2, A_3, A_4 such that

$$\sigma(OA_iA_j) \geq 1 \quad \forall i, j = 1, 2, 3, 4, i \neq j.$$

where $\sigma(OA_iA_j)$ is the area of triangle OA_iA_j. Prove that there exists at least one pair $i_0, j_0 \in \{1, 2, 3, 4\}$ such that

$$\sigma(OA_iA_j) \geq \sqrt{2}.$$

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29 155 birds $P_1, \ldots, P_{155}$ are sitting down on the boundary of a circle C. Two birds P_i, P_j are mutually visible if the angle at centre $m(\cdot)$ of their positions $m(P_iP_j) \leq 10^\circ$. Find the smallest number of mutually visible pairs of birds, i.e. minimal set of pairs $\{x, y\}$ of mutually visible pairs of birds with $x, y \in \{P_1, \ldots, P_{155}\}$. One assumes that a position (point) on C can be occupied simultaneously by several birds, e.g. all possible birds.

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30 Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

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31 Let $a_1 \geq a_2 \geq a_3 \in \mathbb{Z}^+$ be given and let N(a_1, a_2, a_3) be the number of solutions (x_1, x_2, x_3) of the equation

$$\sum_{k=1}^{3} \frac{a_k}{x_k} = 1.$$

where $x_1, x_2,$ and x_3 are positive integers. Prove that

$$N(a_1, a_2, a_3) \leq 6a_1a_2(3 + ln(2a_1)).$$

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32 The vertex A of the acute triangle ABC is equidistant from the circumcenter O and the orthocenter H. Determine all possible values for the measure of angle A.
