IMO Shortlist 1991

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1 Given a point P inside a triangle $\triangle ABC$. Let D, E, F be the orthogonal projections of the point P on the sides BC, CA, AB, respectively. Let the orthogonal projections of the point A on the lines BP and CP be M and N, respectively. Prove that the lines ME, NF, BC are concurrent.

Original formulation:

Let ABC be any triangle and P any point in its interior. Let P_1, P_2 be the feet of the perpendiculars from P to the two sides AC and BC. Draw AP and BP, and from C drop perpendiculars to AP and BP. Let Q_1 and Q_2 be the feet of these perpendiculars. Prove that the lines $Q_1P_2, Q_2P_1,$ and AB are concurrent.

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2 ABC is an acute-angled triangle. M is the midpoint of BC and P is the point on AM such that $MB = MP$. H is the foot of the perpendicular from P to BC. The lines through H perpendicular to PB, PC meet AB, AC respectively at Q, R. Show that BC is tangent to the circle through Q, H, R at H.

Original Formulation:

For an acute triangle ABC, M is the midpoint of the segment BC, P is a point on the segment AM such that $PM = BM, H$ is the foot of the perpendicular line from P to BC, Q is the point of intersection of segment AB and the line passing through H that is perpendicular to $PB,$ and finally, R is the point of intersection of the segment AC and the line passing through H that is perpendicular to $PC.$ Show that the circumcircle of QHR is tangent to the side BC at point $H.$

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3 Let S be any point on the circumscribed circle of $PQR.$ Then the feet of the perpendiculars from S to the three sides of the triangle lie on the same straight line. Denote this line by $l(S, PQR).$ Suppose that the hexagon $ABCDEF$ is inscribed in a circle. Show that the four lines $l(A,BDF),$ $l(B,ACE),$ $l(D,ABF),$ and $l(E,ABC)$ intersect at one point if and only if $CDEF$ is a rectangle.

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4 Let ABC be a triangle and P an interior point of ABC . Show that at least one of the angles $\angle PAB,$ $\angle PBC,$ $\angle PCA$ is less than or equal to 30°.

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5 In the triangle $ABC,$ with $\angle A = 60^{\circ},$ a parallel IF to AC is drawn through the incenter I of the triangle, where F lies on the side $AB.$ The point P on the side BC is such that $3BP = BC.$

Show that $\angle BFP = \frac{\angle B}{2}$.

7 $ABCD$ is a terahedron: $AD+BD = AC+BC$, $BD+CD = BA+CA$, $CD+AD = CB+AB$, M, N, P are the mid points of BC, CA, AB. $OA = OB = OC = OD$. Prove that $\angle MOP = \angle NOP = \angle NOM$.

8 S be a set of n points in the plane. No three points of S are collinear. Prove that there exists a set P containing $2n-5$ points satisfying the following condition: In the interior of every triangle whose three vertices are elements of S lies a point that is an element of P.

9 In the plane we are given a set E of 1991 points, and certain pairs of these points are joined with a path. We suppose that for every point of E, there exist at least 1593 other points of E to which it is joined by a path. Show that there exist six points of E every pair of which are joined by a path.

Alternative version: Is it possible to find a set E of 1991 points in the plane and paths joining certain pairs of the points in E such that every point of E is joined with a path to at least 1592 other points of E, and in every subset of six points of E there exist at least two points that are not joined?

10 Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

Note: Graph-Definition. A **graph** consists of a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices u, v belongs to at most one edge. The graph G is connected if for each pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, \cdots, v_m = y$ such that each pair v_i, v_{i+1} $(0 \leq i < m)$ is joined by an edge of G.

11 Prove that $\sum_{k=0}^{995} \frac{(-1)^k}{1991-k} \binom{1991-k}{k} = \frac{1}{1991}$

12 Let $S = \{1, 2, 3, \cdots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

13 Given any integer $n \geq 2$, assume that the integers $a_1, a_2, \ldots, a_n$ are not divisible by n and, moreover, that n does not divide $\sum_{i=1}^n a_i$. Prove that there exist at least n different sequences $(e_1, e_2, \ldots, e_n)$ consisting of zeros or ones such $\sum_{i=1}^n e_i \cdot a_i$ is divisible by n.

14 Let a, b, c be integers and p an odd prime number. Prove that if $f(x) = ax^2 + bx + c$ is a perfect square for $2p-1$ consecutive integer values of x, then p divides $b^2 - 4ac$.

15 Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence $a_1, a_2, \ldots, a_n, \ldots$ become periodic after a finite number of terms?

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16 Let $n > 6$ be an integer and $a_1, a_2, \cdots, a_k$ be all the natural numbers less than n and relatively prime to n. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

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17 Find all positive integer solutions x, y, z of the equation $3^x + 4^y = 5^z$.

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18 Find the highest degree k of 1991 for which 1991^k divides the number

$$1990^{1991^{1992}} + 1992^{1991^{1990}}.$$

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19 Let α be a rational number with $0 < \alpha < 1$ and $\cos(3\pi\alpha) + 2\cos(2\pi\alpha) = 0$. Prove that $\alpha = \frac{2}{3}$.

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20 Let α be the positive root of the equation $x^2 = 1991x + 1$. For natural numbers m and n define

$$m * n = mn + \lfloor \alpha m \rfloor \lfloor \alpha n \rfloor.$$

Prove that for all natural numbers p, q, and r,

$$(p * q) * r = p * (q * r).$$

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21 Let $f(x)$ be a monic polynomial of degree 1991 with integer coefficients. Define $g(x) = f^2(x) - 9$. Show that the number of distinct integer solutions of $g(x) = 0$ cannot exceed 1995.

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22 Real constants a, b, c are such that there is exactly one square all of whose vertices lie on the cubic curve $y = x^3 + ax^2 + bx + c$. Prove that the square has sides of length $\sqrt[4]{72}$.

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23 Let f and g be two integer-valued functions defined on the set of all integers such that

(a) $f(m + f(f(n))) = -f(f(m+1) - n$ for all integers m and n;
(b) g is a polynomial function with integer coefficients and g(n) = $g(f(n))$ $\forall n \in \mathbb{Z}$.

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24 An odd integer $n \geq 3$ is said to be nice if and only if there is at least one permutation $a_1, \cdots, a_n$ of $1, \cdots, n$ such that the n sums $a_1 - a_2 + a_3 - \cdots - a_{n-1} + a_n$, $a_2 - a_3 + a_3 - \cdots - a_n + a_1$, $a_3 - a_4 + a_5 - \cdots - a_1 + a_2$, $\cdots$, $a_n - a_1 + a_2 - \cdots - a_{n-2} + a_{n-1}$ are all positive. Determine the set of all 'nice' integers.

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25 Suppose that $n \geq 2$ and $x_1, x_2, \dots, x_n$ are real numbers between 0 and 1 (inclusive). Prove that for some index i between 1 and $n-1$ the
inequality

$$x_i(1-x_{i+1}) \geq \frac{1}{4}x_1(1-x_n)$$

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26 Let $n \geq 2, n \in \mathbb{N}$ and let $p, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}$ satisfying $\frac{1}{2} \leq p \leq 1$, $0 \leq a_i$, $0 \leq b_i \leq p$, $i = 1, \dots, n$, and

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i.$$

Prove the inequality:

$$\sum_{i=1}^n b_i \prod_{j=1, j \neq i}^n a_j \leq \frac{p}{(n-1)^{n-1}}.$$

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27 Determine the maximum value of the sum

$$\sum_{i<j} x_i x_j (x_i + x_j)$$

over all $n-$tuples $(x_1, \dots, x_n)$, satisfying $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1.$

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28 An infinite sequence $x_0, x_1, x_2, \dots$ of real numbers is said to be **bounded** if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$. Given any real number $a > 1,$ construct a bounded infinite sequence $x_0, x_1, x_2, \dots$ such that

$$|x_i - x_j||i-j|^a \geq 1$$

for every pair of distinct nonnegative integers i, j.

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29 We call a set S on the real line $\mathbb{R}$ *superinvariant* if for any stretching A of the set by the transformation taking x to $A(x) = x_0 + a(x - x_0), a > 0$ there exists a translation B, $B(x) = x + b$, such that the images of S under A and B agree; i.e., for any $x \in S$ there is a $y \in S$ such that $A(x) = B(y)$ and for any $t \in S$ there is a $u \in S$ such that $B(t) = A(u)$. Determine all *superinvariant* sets.

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30 Two students A and B are playing the following game: Each of them writes down on a sheet of paper a positive integer and gives the sheet to the referee. The referee writes down on a blackboard two integers, one of which is the sum of the integers written by the players. After that, the referee asks student A : Can you tell the integer written by the other student? If A answers no, the referee puts the same question to student B. If B answers no, the referee puts the question back to $A,$ and so on. Assume that both students are intelligent and truthful. Prove that after a finite number of questions, one of the students will answer yes.