IMO Shortlist 1992

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1 Prove that for any positive integer m there exist an infinite number of pairs of integers (x, y) such that

(i) x and y are relatively prime;
(ii) y divides $x^2 + m$;
(iii) x divides $y^2 + m$.
(iv) $x + y \leq m + 1-$ (optional condition)

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2 Let $\mathbb{R}^+$ be the set of all non-negative real numbers. Given two positive real numbers a and b, suppose that a mapping $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfies the functional equation:

$$f(f(x)) + af(x) = b(a + b)x.$$

Prove that there exists a unique solution of this equation.

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3 The diagonals of a quadrilateral $ABCD$ are perpendicular: $AC \perp BD$. Four squares, $ABEF, BCGH, CDIJ$ are erected externally on its sides. The intersection points of the pairs of straight lines CL, DF, AH, BJ are denoted by P_1, Q_1, R_1, S_1, respectively (left figure), and the intersection points of the pairs of straight lines $AI, BK, CEDG$ are denoted by P_2, Q_2, R_2, S_2, respectively (right figure). Prove that $P_1Q_1R_1S_1 \cong P_2Q_2R_2S_2$ where P_1, Q_1, R_1, S_1 and P_2, Q_2, R_2, S_2 are the two quadrilaterals.

Alternative formulation: Outside a convex quadrilateral $ABCD$ with perpendicular diagonals, four squares $AEFB, BGHC, CIJD, DKLA$, are constructed (vertices are given in counter-clockwise order). Prove that the quadrilaterals Q_1 and Q_2 formed by the lines AG, BI, CK, DE and AJ, BL, CF, DH, respectively, are congruent.

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4 Consider 9 points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

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5 A convex quadrilateral has equal diagonals. An equilateral triangle is constructed on the outside of each side of the quadrilateral. The centers of the triangles on opposite sides are joined. Show that the two joining lines are perpendicular.

Alternative formulation. Given a convex quadrilateral $ABCD$ with congruent diagonals $AC =$

BD. Four regular triangles are errected externally on its sides. Prove that the segments joining the centroids of the triangles on the opposite sides are perpendicular to each other.

Original formulation: Let $ABCD$ be a convex quadrilateral such that $AC = BD$. Equilateral triangles are constructed on the sides of the quadrilateral. Let O_1, O_2, O_3, O_4 be the centers of the triangles constructed on AB, BC, CD, DA respectively. Show that O_1O_3 is perpendicular to O_2O_4.

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6 Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^2 + f(y)\right) = y + (f(x))^2 \quad \text{for all } x, y \in \mathbb{R}.$$

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7 Two circles Ω_1 and Ω_2 are externally tangent to each other at a point I, and both of these circles are tangent to a third circle Ω which encloses the two circles Ω_1 and Ω_2.
The common tangent to the two circles Ω_1 and Ω_2 at the point I meets the circle Ω at a point A.
One common tangent to the circles Ω_1 and Ω_2 which doesn't pass through I meets the circle Ω at the points B and C such that the points A and I lie on the same side of the line BC.
Prove that the point I is the incenter of triangle ABC.

Alternative formulation. Two circles touch externally at a point I. The two circles lie inside a large circle and both touch it. The chord BC of the large circle touches both smaller circles (not at I). The common tangent to the two smaller circles at the point I meets the large circle at a point A, where the points A and I are on the same side of the chord BC. Show that the point I is the incenter of triangle ABC.

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8 Show that in the plane there exists a convex polygon of 1992 sides satisfying the following conditions:

(i) its side lengths are $1, 2, 3, \ldots, 1992$ in some order;
(ii) the polygon is circumscribable about a circle.

Alternative formulation: Does there exist a 1992-gon with side lengths $1, 2, 3, \ldots, 1992$ circumscribed about a circle? Answer the same question for a 1990-gon.

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9 Let $f(x)$ be a polynomial with rational coefficients and α be a real number such that

$$\alpha^3 - \alpha = [f(\alpha)]^3 - f(\alpha) = 33^{1992}.$$

Prove that for each $n \geq 1$,

$$[f^n(\alpha)]^3 - f^n(\alpha) = 33^{1992},$$

where $f^n(x) = f(f(\cdots f(x)))$, and n is a positive integer.

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10 Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set A.

Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.

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11 In a triangle ABC, let D and E be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with the sides AC, AB, respectively. Determine the angles $\angle A, \angle B, \angle C$ if $\angle BDE = 24^\circ$, $\angle CED = 18^\circ$.

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12 Let f, g and a be polynomials with real coefficients, f and g in one variable and a in two variables. Suppose

$$f(x) - f(y) = a(x, y)(g(x) - g(y)) \forall x, y \in \mathbb{R}$$

Prove that there exists a polynomial h with $f(x) = h(g(x))\ \forall x \in \mathbb{R}$.

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13 Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1)$$

is a divisor of $abc - 1$.

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14 For any positive integer x define $g(x)$ as greatest odd divisor of x, and

$$f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)} & \text{if } x \text{ is even,} \\ 2^{\frac{x+1}{2}} & \text{if } x \text{ is odd.} \end{cases}$$

Construct the sequence $x_1 = 1, x_{n+1} = f(x_n)$. Show that the number 1992 appears in this sequence, determine the least n such that $x_n = 1992$, and determine whether n is unique.

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15 Does there exist a set M with the following properties?

(i) The set M consists of 1992 natural numbers.
(ii) Every element in M and the sum of any number of elements have the form m^k $(m, k \in \mathbb{N}, k \geq 2)$.

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16 Prove that $\frac{5^{125}-1}{5^{25}-1}$ is a composite number.

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17 Let $\alpha(n)$ be the number of digits equal to one in the binary representation of a positive integer n. Prove that:

(a) the inequality $\alpha(n)(n^2) \leq \frac{1}{2}\alpha(n)(\alpha(n)+1)$ holds;
(b) the above inequality is an equality for infinitely many positive integers, and
(c) there exists a sequence $(n_i)_1^\infty$ such that $\frac{\alpha(n_i^2)}{\alpha(n_i}$
goes to zero as i goes to ∞.

Alternative problem: Prove that there exists a sequence a sequence $(n_i)_1^\infty$ such that $\frac{\alpha(n_i^2)}{\alpha(n_i)}$

(d) ∞;
(e) an arbitrary real number $\gamma \in (0,1)$;
(f) an arbitrary real number $\gamma \geq 0$;

as i goes to ∞.

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18 Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Pick any x_1 in $[0,1)$ and define the sequence $x_1, x_2, x_3, \ldots$ by $x_{n+1} = 0$ if $x_n = 0$ and $x_{n+1} = \frac{1}{x_n} - \left\lfloor \frac{1}{x_n} \right\rfloor$ otherwise. Prove that

$$x_1 + x_2 + \ldots + x_n < \frac{F_1}{F_2} + \frac{F_2}{F_3} + \ldots + \frac{F_n}{F_{n+1}},$$

where $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

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19 Let $f(x) = x^8 + 4x^6 + 2x^4 + 28x^2 + 1$. Let $p > 3$ be a prime and suppose there exists an integer z such that p divides $f(z)$. Prove that there exist integers $z_1, z_2, \ldots, z_8$ such that if

$$g(x) = (x - z_1)(x - z_2) \cdot \ldots \cdot (x - z_8),$$

then all coefficients of $f(x) - g(x)$ are divisible by p.

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20 In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

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21 For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.

a.) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
b.) Find an integer n such that $S(n) = n^2 - 14$.
c.) Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.
