

**IMO Shortlist 1996**
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## – Algebra

**1** Suppose that  $a, b, c > 0$  such that  $abc = 1$ . Prove that

$$\frac{ab}{ab + a^5 + b^5} + \frac{bc}{bc + b^5 + c^5} + \frac{ca}{ca + c^5 + a^5} \leq 1.$$

**2** Let  $a_1 \geq a_2 \geq \dots \geq a_n$  be real numbers such that for all integers  $k > 0$ ,

$$a_1^k + a_2^k + \dots + a_n^k \geq 0.$$

 Let  $p = \max\{|a_1|, \dots, |a_n|\}$ . Prove that  $p = a_1$  and that

$$(x - a_1) \cdot (x - a_2) \cdot \dots \cdot (x - a_n) \leq x^n - a_1^n$$

 for all  $x > a_1$ .

**3** Let  $a > 2$  be given, and starting  $a_0 = 1, a_1 = a$  define recursively:

$$a_{n+1} = \left( \frac{a_n^2}{a_{n-1}^2} - 2 \right) \cdot a_n.$$

 Show that for all integers  $k > 0$ , we have:  $\sum_{i=0}^k \frac{1}{a_i} < \frac{1}{2} \cdot (2 + a - \sqrt{a^2 - 4})$ .

**4** Let  $a_1, a_2, \dots, a_n$  be non-negative reals, not all zero. Show that that

- (a) The polynomial  $p(x) = x^n - a_1 x^{n-1} + \dots - a_{n-1} x - a_n$  has precisely 1 positive real root  $R$ .  
 (b) let  $A = \sum_{i=1}^n a_i$  and  $B = \sum_{i=1}^n i a_i$ . Show that  $A^A \leq R^B$ .

**5** Let  $P(x)$  be the real polynomial function,  $P(x) = ax^3 + bx^2 + cx + d$ . Prove that if  $|P(x)| \leq 1$  for all  $x$  such that  $|x| \leq 1$ , then

$$|a| + |b| + |c| + |d| \leq 7.$$

- 6 Let  $n$  be an even positive integer. Prove that there exists a positive integer  $k$  such that

$$k = f(x) \cdot (x+1)^n + g(x) \cdot (x^n + 1)$$

for some polynomials  $f(x), g(x)$  having integer coefficients. If  $k_0$  denotes the least such  $k$ , determine  $k_0$  as a function of  $n$ , i.e. show that  $k_0 = 2^q$  where  $q$  is the odd integer determined by  $n = q \cdot 2^r, r \in \mathbb{N}$ .

Note: This is variant A6' of the three variants given for this problem.

- 7 Let  $f$  be a function from the set of real numbers  $\mathbb{R}$  into itself such for all  $x \in \mathbb{R}$ , we have  $|f(x)| \leq 1$  and

$$f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right).$$

Prove that  $f$  is a periodic function (that is, there exists a non-zero real number  $c$  such  $f(x+c) = f(x)$  for all  $x \in \mathbb{R}$ ).

- 8 Let  $\mathbb{N}_0$  denote the set of nonnegative integers. Find all functions  $f$  from  $\mathbb{N}_0$  to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \quad \text{for all } m, n \in \mathbb{N}_0.$$

- 9 Let the sequence  $a(n), n = 1, 2, 3, \dots$  be generated as follows with  $a(1) = 0$ , and for  $n > 1$ :

$$a(n) = a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + (-1)^{\frac{n(n+1)}{2}}.$$

1.) Determine the maximum and minimum value of  $a(n)$  over  $n \leq 1996$  and find all  $n \leq 1996$  for which these extreme values are attained.

2.) How many terms  $a(n), n \leq 1996$ , are equal to 0?

– Geometry

- 1 Let  $ABC$  be a triangle, and  $H$  its orthocenter. Let  $P$  be a point on the circumcircle of triangle  $ABC$  (distinct from the vertices  $A, B, C$ ), and let  $E$  be the foot of the altitude of triangle  $ABC$  from the vertex  $B$ . Let the parallel to the line  $BP$  through the point  $A$  meet the parallel to the line  $AP$  through the point  $B$  at a point  $Q$ . Let the parallel to the line  $CP$  through the point  $A$  meet the parallel to the line  $AP$  through the point  $C$  at a point  $R$ . The lines  $HR$  and  $AQ$  intersect at some point  $X$ . Prove that the lines  $EX$  and  $AP$  are parallel.

- 2 Let  $P$  be a point inside a triangle  $ABC$  such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let  $D, E$  be the incenters of triangles  $APB, APC$ , respectively. Show that the lines  $AP, BD, CE$  meet at a point.

- 3 Let  $O$  be the circumcenter and  $H$  the orthocenter of an acute-angled triangle  $ABC$  such that  $BC > CA$ . Let  $F$  be the foot of the altitude  $CH$  of triangle  $ABC$ . The perpendicular to the line  $OF$  at the point  $F$  intersects the line  $AC$  at  $P$ . Prove that  $\angle FHP = \angle BAC$ .

- 4 Let  $ABC$  be an equilateral triangle and let  $P$  be a point in its interior. Let the lines  $AP, BP, CP$  meet the sides  $BC, CA, AB$  at the points  $A_1, B_1, C_1$ , respectively. Prove that  $A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A$ .

- 5 Let  $ABCDEF$  be a convex hexagon such that  $AB$  is parallel to  $DE$ ,  $BC$  is parallel to  $EF$ , and  $CD$  is parallel to  $FA$ . Let  $R_A, R_C, R_E$  denote the circumradii of triangles  $FAB, BCD, DEF$ , respectively, and let  $P$  denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

- 6 Let the sides of two rectangles be  $\{a, b\}$  and  $\{c, d\}$ , respectively, with  $a < c \leq d < b$  and  $ab < cd$ . Prove that the first rectangle can be placed within the second one if and only if

$$(b^2 - a^2)^2 \leq (bc - ad)^2 + (bd - ac)^2.$$

- 7 Let  $ABC$  be an acute triangle with circumcenter  $O$  and circumradius  $R$ .  $AO$  meets the circumcircle of  $BOC$  at  $A'$ ,  $BO$  meets the circumcircle of  $COA$  at  $B'$  and  $CO$  meets the circumcircle of  $AOB$  at  $C'$ . Prove that

$$OA' \cdot OB' \cdot OC' \geq 8R^3.$$

Sorry if this has been posted before since this is a very classical problem, but I failed to find it with the search-function.

- 8 Let  $ABCD$  be a convex quadrilateral, and let  $R_A, R_B, R_C, R_D$  denote the circumradii of the triangles  $DAB, ABC, BCD, CDA$ , respectively. Prove that  $R_A + R_C > R_B + R_D$  if and only if  $\angle A + \angle C > \angle B + \angle D$ .

- 9 In the plane, consider a point  $X$  and a polygon  $\mathcal{F}$  (which is not necessarily convex). Let  $p$  denote the perimeter of  $\mathcal{F}$ , let  $d$  be the sum of the distances from the point  $X$  to the vertices of  $\mathcal{F}$ , and let  $h$  be the sum of the distances from the point  $X$  to the sidelines of  $\mathcal{F}$ . Prove that  $d^2 - h^2 \geq \frac{p^2}{4}$ .

## – Number Theory

**1** Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle, moving in a clockwise direction; that is, the numbers  $a, b, c, d$  are replaced by  $a - b, b - c, c - d, d - a$ . Is it possible after 1996 such to have numbers  $a, b, c, d$  such the numbers  $|bc - ad|, |ac - bd|, |ab - cd|$  are primes?

**2** The positive integers  $a$  and  $b$  are such that the numbers  $15a + 16b$  and  $16a - 15b$  are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

**3** A finite sequence of integers  $a_0, a_1, \dots, a_n$  is called quadratic if for each  $i$  in the set  $\{1, 2, \dots, n\}$  we have the equality  $|a_i - a_{i-1}| = i^2$ .

a.) Prove that any two integers  $b$  and  $c$ , there exists a natural number  $n$  and a quadratic sequence with  $a_0 = b$  and  $a_n = c$ .

b.) Find the smallest natural number  $n$  for which there exists a quadratic sequence with  $a_0 = 0$  and  $a_n = 1996$ .

**4** Find all positive integers  $a$  and  $b$  for which

$$\left\lfloor \frac{a^2}{b} \right\rfloor + \left\lfloor \frac{b^2}{a} \right\rfloor = \left\lfloor \frac{a^2 + b^2}{ab} \right\rfloor + ab.$$

**5** Show that there exists a bijective function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that for all  $m, n \in \mathbb{N}_0$ :

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

## – Combinatorics

**1** We are given a positive integer  $r$  and a rectangular board  $ABCD$  with dimensions  $AB = 20, BC = 12$ . The rectangle is divided into a grid of  $20 \times 12$  unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is  $\sqrt{r}$ . The task is to find a sequence of moves leading from the square with  $A$  as a vertex to the square with  $B$  as a vertex.

(a) Show that the task cannot be done if  $r$  is divisible by 2 or 3.

(b) Prove that the task is possible when  $r = 73$ .

(c) Can the task be done when  $r = 97$ ?

**2** A square  $(n - 1) \times (n - 1)$  is divided into  $(n - 1)^2$  unit squares in the usual manner. Each of the  $n^2$  vertices of these squares is to be coloured red or blue. Find the number of different colourings such that each unit square has exactly two red vertices. (Two colouring schemes are regarded as different if at least one vertex is coloured differently in the two schemes.)

**3** Let  $k, m, n$  be integers such that  $1 < n \leq m - 1 \leq k$ . Determine the maximum size of a subset  $S$  of the set  $\{1, 2, 3, \dots, k - 1, k\}$  such that no  $n$  distinct elements of  $S$  add up to  $m$ .

**4** Determine whether or not there exist two disjoint infinite sets  $A$  and  $B$  of points in the plane satisfying the following conditions:

a.) No three points in  $A \cup B$  are collinear, and the distance between any two points in  $A \cup B$  is at least 1.

b.) There is a point of  $A$  in any triangle whose vertices are in  $B$ , and there is a point of  $B$  in any triangle whose vertices are in  $A$ .

**5** Let  $p, q, n$  be three positive integers with  $p + q < n$ . Let  $(x_0, x_1, \dots, x_n)$  be an  $(n + 1)$ -tuple of integers satisfying the following conditions :

(a)  $x_0 = x_n = 0$ , and

(b) For each  $i$  with  $1 \leq i \leq n$ , either  $x_i - x_{i-1} = p$  or  $x_i - x_{i-1} = -q$ .

Show that there exist indices  $i < j$  with  $(i, j) \neq (0, n)$ , such that  $x_i = x_j$ .

**6** A finite number of coins are placed on an infinite row of squares. A sequence of moves is performed as follows: at each stage a square containing more than one coin is chosen. Two coins are taken from this square; one of them is placed on the square immediately to the left while the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one coin on each square. Given some initial configuration, show that any legal sequence of moves will terminate after the same number of steps and with the same final configuration.

**7** let  $V$  be a finite set and  $g$  and  $f$  be two injective surjective functions from  $V$  to  $V$ . let  $T$  and  $S$  be two sets such that they are defined as following"  $S = \{w \in V : f(f(w)) = g(g(w))\}$   
 $T = \{w \in V : f(g(w)) = g(f(w))\}$   
 we know that  $S \cup T = V$ , prove:  
 for each  $w \in V : f(w) \in S$  if and only if  $g(w) \in S$

