IMO Shortlist 1997

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1 In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

b) Prove that $f(m,n) \leq \frac{1}{2}\max\{m,n\}$ for all m and n.

c) Show that there is no constant $C \in \mathbb{R}$ such that $f(m,n) < C$ for all m and n.

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2 Let $R_1, R_2, \ldots$ be the family of finite sequences of positive integers defined by the following rules: $R_1 = (1)$, and if $R_{n-1} = (x_1, \ldots, x_s)$, then

$$R_n = (1, 2, \ldots, x_1, 1, 2, \ldots, x_2, \ldots, 1, 2, \ldots, x_s, n).$$

For example, $R_2 = (1,2)$, $R_3 = (1,1,2,3)$, $R_4 = (1,1,1,2,1,2,3,4)$. Prove that if $n > 1$, then the kth term from the left in R_n is equal to 1 if and only if the kth term from the right in R_n is different from 1.

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3 For each finite set U of nonzero vectors in the plane we define $l(U)$ to be the length of the vector that is the sum of all vectors in $U.$ Given a finite set V of nonzero vectors in the plane, a subset B of V is said to be maximal if $l(B)$ is greater than or equal to $l(A)$ for each nonempty subset A of $V.$

(a) Construct sets of 4 and 5 vectors that have 8 and 10 maximal subsets respectively.

(b) Show that, for any set V consisting of $n \geq 1$ vectors the number of maximal subsets is less than or equal to $2n.$

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4 An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver matrix* if, for each $i = 1, 2, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

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5 Let $ABCD$ be a regular tetrahedron and M, N distinct points in the planes ABC and ADC respectively. Show that the segments MN, BN, MD are the sides of a triangle.

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6 (a) Let n be a positive integer. Prove that there exist distinct positive integers x, y, z such that

$$x^{n-1} + y^n = z^{n+1}.$$

(b) Let a, b, c be positive integers such that a and b are relatively prime and c is relatively prime either to a or to b. Prove that there exist infinitely many triples (x, y, z) of distinct positive integers x, y, z such that

$$x^a + y^b = z^c.$$

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7 The lengths of the sides of a convex hexagon $ABCDEF$ satisfy $AB = BC$, $CD = DE$, $EF = FA$. Prove that:

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

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8 It is known that $\angle BAC$ is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T.

Show that $AU = TB + TC$.

Alternative formulation:

Four different points A, B, C, D are chosen on a circle Γ such that the triangle BCD is not right-angled. Prove that:

(a) The perpendicular bisectors of AB and AC meet the line AD at certain points W and V, respectively, and that the lines CV and BW meet at a certain point T.

(b) The length of one of the line segments $AD, BT,$ and CT is the sum of the lengths of the other two.

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9 Let $A_1A_2A_3$ be a non-isosceles triangle with incenter I. Let C_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (the addition of indices being mod 3). Let $B_i, i = 1, 2, 3,$ be the second point of intersection of C_{i+1} and C_{i+2}. Prove that the circumcentres of the triangles $A_1B_1I, A_2B_2I, A_3B_3I$ are collinear.

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10 Find all positive integers k for which the following statement is true: If $F(x)$ is a polynomial with integer coefficients satisfying the condition $0 \leq F(c) \leq k$ for each $c \in \{0, 1, \ldots, k+1\}$, then $F(0) = F(1) = \ldots = F(k+1)$.

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11 Let $P(x)$ be a polynomial with real coefficients such that $P(x) > 0$ for all $x \geq 0$. Prove that there exists a positive integer n such that $(1+x)^n \cdot P(x)$ is a polynomial with nonnegative coefficients.

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12 Let p be a prime number and f an integer polynomial of degree d such that $f(0) = 0, f(1) = 1$ and $f(n)$ is congruent to 0 or 1 modulo p for every integer n. Prove that $d \geq p-1$.

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13 In town A, there are n girls and n boys, and each girl knows each boy. In town B, there are n girls $g_1, g_2, \ldots, g_n$ and $2n-1$ boys $b_1, b_2, \ldots, b_{2n-1}$. The girl g_i, $i = 1, 2, \ldots, n$, knows the boys $b_1, b_2, \ldots, b_{2i-1}$, and no others. For all $r = 1, 2, \ldots, n$, denote by $A(r), B(r)$ the number of different ways in which r girls from town A, respectively town B, can dance with r boys from their own town, forming r pairs, each girl with a boy she knows. Prove that $A(r) = B(r)$ for each $r = 1, 2, \ldots, n$.

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14 Let b, m, n be positive integers such that $b > 1$ and $m \neq n$. Prove that if $b^m - 1$ and $b^n - 1$ have the same prime divisors, then $b+1$ is a power of 2.

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15 An infinite arithmetic progression whose terms are positive integers contains the square of an integer and the cube of an integer. Show that it contains the sixth power of an integer.

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16 In an acute-angled triangle ABC, let AD, BE be altitudes and AP, BQ internal bisectors. Denote by I and O the incenter and the circumcentre of the triangle, respectively. Prove that the points D, E, and I are collinear if and only if the points P, Q, and O are collinear.

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17 Find all pairs (a, b) of positive integers that satisfy the equation: $a^{b^2} = b^a$.

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18 The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F, respectively. The line through D parallel to EF meets the lines AC and AB at Q and R, respectively. The line EF meets BC at P. Prove that the circumcircle of the triangle PQR passes through the midpoint of BC.

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19 Let $a_1 \geq \cdots \geq a_n \geq a_{n+1} = 0$ be real numbers. Show that

$$\sqrt{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n} \sqrt{k}(\sqrt{a_k} - \sqrt{a_{k+1}}).$$

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20 Let ABC be a triangle. D is a point on the side (BC). The line AD meets the circumcircle again at X. P is the foot of the perpendicular from X to AB, and Q is the foot of the perpendicular from X to AC. Show that the line PQ is a tangent to the circle on diameter XD if and only if $AB = AC$.

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21 Let x_1, x_2, ..., x_n be real numbers satisfying the conditions:

$$\begin{cases} |x_1 + x_2 + \cdots + x_n| & = & 1 \\ |x_i| & \leq & \dfrac{n+1}{2} \quad \text{for } i = 1, 2, \ldots, n. \end{cases}$$

Show that there exists a permutation y_1, y_2, ..., y_n of x_1, x_2, ..., x_n such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

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22 Does there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^k$ for all real numbers x

a) if $k = 3$?

b) if $k = 4$?

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23 Let $ABCD$ be a convex quadrilateral. The diagonals AC and BD intersect at K. Show that $ABCD$ is cyclic if and only if $AK \sin A + CK \sin C = BK \sin B + DK \sin D$.

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24 For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways: 4; 2+2; 2+1+1; 1+1+1+1.

Prove that, for any integer $n \geq 3$ we have $2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}$.

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25 Let X, Y, Z be the midpoints of the small arcs BC, CA, AB respectively (arcs of the circumcircle of ABC). M is an arbitrary point on BC, and the parallels through M to the internal bisectors of $\angle B, \angle C$ cut the external bisectors of $\angle C, \angle B$ in N, P respectively. Show that XM, YN, ZP concur.

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26 For every integer $n \geq 2$ determine the minimum value that the sum $\sum_{i=0}^{n} a_i$ can take for nonnegative numbers $a_0, a_1, \ldots, a_n$ satisfying the condition $a_0 = 1$, $a_i \leq a_{i+1} + a_{i+2}$ for $i = 0, \ldots, n-2$.
