7th RMM 2015

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Day 1 February 27, 2015

1 Does there exist an infinite sequence of positive integers $a_1, a_2, a_3, \dots$ such that a_m and a_n are coprime if and only if $|m-n|=1$?

2 For an integer $n \geq 5,$ two players play the following game on a regular n-gon. Initially, three consecutive vertices are chosen, and one counter is placed on each. A move consists of one player sliding one counter along any number of edges to another vertex of the n-gon without jumping over another counter. A move is legal if the area of the triangle formed by the counters is strictly greater after the move than before. The players take turns to make legal moves, and if a player cannot make a legal move, that player loses. For which values of n does the player making the first move have a winning strategy?

3 A finite list of rational numbers is written on a blackboard. In an *operation*, we choose any two numbers a, b, erase them, and write down one of the numbers

$$a+b,\ a-b,\ b-a,\ a\times b,\ a/b \text{ (if } b\neq 0),\ b/a \text{ (if } a\neq 0).$$

Prove that, for every integer $n>100$, there are only finitely many integers $k\geq 0$, such that, starting from the list

$$k+1,\ k+2,\ \dots,\ k+n,$$

it is possible to obtain, after $n-1$ operations, the value $n!$.

Day 2 February 28, 2015

4 Let ABC be a triangle, and let D be the point where the incircle meets side BC. Let J_b and J_c be the incentres of the triangles ABD and ACD, respectively. Prove that the circumcentre of the triangle AJ_bJ_c lies on the angle bisector of $\angle BAC$.

5 Let $p\geq 5$ be a prime number. For a positive integer k, let $R(k)$ be the remainder when k is divided by p, with $0\leq R(k)\leq p-1$. Determine all positive integers $a<p$ such that, for every $m=1,2,\cdots,p-1$,

$$m+R(ma)>a.$$

6 Given a positive integer n, determine the largest real number μ satisfying the following condition: for every set C of $4n$ points in the interior of the unit square U, there exists a rectangle T contained in U such that

- the sides of T are parallel to the sides of U;
- the interior of T contains exactly one point of C;
- the area of T is at least μ.