International Zhautykov Olympiad 2017

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Day 1

1 Let ABC be a non-isosceles triangle with circumcircle ω and let H, M be orthocenter and midpoint of AB respectively. Let P, Q be points on the arc AB of ω not containing C such that $\angle ACP = \angle BCQ < \angle ACQ$.Let R, S be the foot of altitudes from H to CQ, CP respectively. Prove that th points P, Q, R, S are concyclic and M is the center of this circle.

2 Find all functions $f : R \to R$ such that

$$(x + y^2)f(yf(x)) = xyf(y^2 + f(x))$$

, where $x, y \in \mathbb{R}$

3 Rectangle on a checked paper with length of a unit square side being 1 Is divided into domino figures(two unit square sharing a common edge). Prove that you colour all corners of squares on the edge of rectangle and inside rectangle with 3 colours such that for any two corners with distance 1 the following conditions hold: they are coloured in different colour if the line connecting the two corners is on the border of two domino figures and coloured in same colour if the line connecting the two corners is inside a domino figure.

Day 2

1 Let (a_n) be sequnce of positive integers such that first k members $a_1, a_2, ..., a_k$ are distinct positive integers, and for each $n > k$, number a_n is the smallest positive integer that can't be represented as a sum of several (possibly one) of the numbers $a_1, a_2, ..., a_{n-1}$. Prove that $a_n = 2a_{n-1}$ for all sufficently large n.

2 For each positive integer k denote $C(k)$ to be sum of its distinct prime divisors. For example $C(1) = 0, C(2) = 2, C(45) = 8$. Find all positive integers n for which $C(2^n + 1) = C(n)$.

3 Let $ABCD$ be the regular tetrahedron, and M, N points in space. Prove that: $AM \cdot AN + BM \cdot BN + CM \cdot CN \geq DM \cdot DN$