IMO Shortlist 2006

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by mattilgale, malinger, pohoatza, Davron, ZetaX, iura, Igor, Valentin Vornicu, e.lopes, cancer

– Algebra

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1 A sequence of real numbers $a_0,\ a_1,\ a_2,\dots$ is defined by the formula

$$a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle \qquad \text{for} \quad i \geq 0;$$

here a_0 is an arbitrary real number, $\lfloor a_i \rfloor$ denotes the greatest integer not exceeding a_i, and $\langle a_i \rangle = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for i sufficiently large.

Proposed by Harmel Nestra, Estionia

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2 The sequence of real numbers $a_0, a_1, a_2, \dots$ is defined recursively by

$$a_0 = -1, \qquad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \quad \text{for} \quad n \geq 1.$$

Show that $a_n > 0$ for all $n \geq 1$.

Proposed by Mariusz Skalba, Poland

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3 The sequence $c_0, c_1, ..., c_n, ...$ is defined by $c_0 = 1, c_1 = 0$, and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set S of ordered pairs (x, y) for which there is a finite set J of positive integers such that $x = \sum_{j \in J} c_j$, $y = \sum_{j \in J} c_{j-1}$. Prove that there exist real numbers α, β, and M with the following property: An ordered pair of nonnegative integers (x, y) satisfies the inequality

$$m < \alpha x + \beta y < M$$

if and only if $(x, y) \in S$.

Remark: A sum over the elements of the empty set is assumed to be 0.

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4 Prove the inequality:

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \cdot \sum_{i<j} a_i a_j$$

for positive reals $a_1, a_2, \dots, a_n$.

Proposed by Dusan Dukic, Serbia

5 If a, b, c are the sides of a triangle, prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}\leq 3$$

Proposed by Hojoo Lee, Korea

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6 Determine the least real number M such that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)|\leq M(a^2+b^2+c^2)^2$$

holds for all real numbers a, b and c.

– Combinatorics

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1 We have $n \geq 2$ lamps $L_1, ..., L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off; otherwise, L_i is switched on.
Initially all the lamps are off except the leftmost one which is on.
(a) Prove that there are infinitely many integers n for which all the lamps will eventually be off.
(b) Prove that there are infinitely many integers n for which the lamps will never be all off.

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2 Let P be a regular 2006-gon. A diagonal is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.
Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

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3 Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number x

$$\sum_P x^{a(P)}(1-x)^{b(P)}=1,$$

where the sum is taken over all convex polygons with vertices in S.

Alternative formulation:

Let M be a finite point set in the plane and no three points are collinear. A subset A of M will be called round if its elements is the set of vertices of a convex $A-$gon $V(A)$. For each round subset let $r(A)$ be the number of points from M which are exterior from the convex $A-$gon

$V(A)$. Subsets with $0, 1$ and 2 elements are always round, its corresponding polygons are the empty set, a point or a segment, respectively (for which all other points that are not vertices of the polygon are exterior). For each round subset A of M construct the polynomial

$$P_A(x) = x^{|A|}(1-x)^{r(A)}.$$

Show that the sum of polynomials for all round subsets is exactly the polynomial $P(x) = 1$.

Proposed by Federico Ardila, Colombia

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4 A cake has the form of an n x n square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of switches, defined as follows:

A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

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5 An $(n,k)-$ tournament is a contest with n players held in k rounds such that:

(i) Each player plays in each round, and every two players meet at most once. (ii) If player A meets player B in round i, player C meets player D in round i, and player A meets player C in round j, then player B meets player D in round j.

Determine all pairs (n,k) for which there exists an $(n,k)-$ tournament.

Proposed by Carlos di Fiore, Argentina

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6 A holey triangle is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A diamond is a $60^\circ - 120^\circ$ unit rhombus.
Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \leq k \leq n$.

Proposed by Federico Ardila, Colombia

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7 Consider a convex polyhedron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it. Call a pair of points of the polyhedron *antipodal* if there exist two parallel planes passing through these points and such that the polyhedron is contained between these planes. Let A be the number of antipodal pairs of vertices, and let B be the number of antipodal pairs of midpoint edges. Determine the difference $A-B$ in terms of the numbers of vertices, edges, and faces.

Proposed by Kei Irei, Japan

– Geometry

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1 Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

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2 Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points P, Q, B and C are concyclic.

Proposed by Vyacheslev Yasinskiy, Ukraine

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3 Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

Proposed by Zuming Feng, USA

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4 A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.

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5 In triangle ABC, let J be the center of the excircle tangent to side BC at A_1 and to the extensions of the sides AC and AB at B_1 and C_1 respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

Proposed by Dimitris Kontogiannis, Greece

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6 Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

7 In a triangle ABC, let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively, and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the vertices A, B, C, respectively. For $i \in \{a, b, c\}$, let w_i be the circle with M_iT_i as diameter. Let p_i be the common external common tangent to the circles w_j and w_k (for all $\{i, j, k\} = \{a, b, c\}$) such that w_i lies on the opposite side of p_i than w_j and w_k do.
Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.

Proposed by Tomas Jurik, Slovakia

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8 Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that

$$\angle PAB + \angle PDC \leq 90^\circ \qquad \text{and} \qquad \angle PBA + \angle PCD \leq 90^\circ.$$

Prove that $AB + CD \geq BC + AD$.

Proposed by Waldemar Pompe, Poland

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9 Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

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10 Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

– Number Theory

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1 Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

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2 For $x \in (0, 1)$ let $y \in (0, 1)$ be the number whose n-th digit after the decimal point is the 2^n-th digit after the decimal point of x. Show that if x is rational then so is y.

Proposed by J.P. Grossman, Canada

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3 We define a sequence $(a_1, a_2, a_3, \ldots)$ by

$$a_n = \frac{1}{n}\left(\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor\right),$$

where $\lfloor x \rfloor$ denotes the integer part of x.

a) Prove that $a_{n+1} > a_n$ infinitely often.
b) Prove that $a_{n+1} < a_n$ infinitely often.

Proposed by Johan Meyer, South Africa

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4 Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

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5 Find all integer solutions of the equation

$$\frac{x^7 - 1}{x - 1} = y^5 - 1.$$

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6 Let $a > b > 1$ be relatively prime positive integers. Define the weight of an integer c, denoted by $w(c)$ to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers x and y such that

$$ax + by = c.$$

An integer c is called a *local champion* if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$.

Find all local champions and determine their number.

Proposed by Zoran Sunic, USA

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7 For all positive integers n, show that there exists a positive integer m such that n divides $2^m + m$.

Proposed by Juhan Aru, Estonia