Mathematical Olympiad 2016

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– Time: 4.5 hours. Each problem is worth 8 points.

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1 The operations below can be applied on any expression of the form ax^2+bx+c.

(I) If $c \neq 0$, replace a by $4a-\frac{3}{c}$ and c by $\frac{c}{4}$.

(II) If $a \neq 0$, replace a by $-\frac{a}{2}$ and c by $-2c+\frac{3}{a}$.

(III$_t$) Replace x by $x-t$, where t is an integer. (Different values of t can be used.)

Is it possible to transform x^2-x-6 into each of the following by applying some sequence of the above operations?

(a) $5x^2+5x-1$

(b) x^2+6x+2

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2 Prove that the arithmetic sequence $5, 11, 17, 23, 29, \dots$ contains infinitely many primes.

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3 Let n be any positive integer. Prove that

$$\sum_{i=1}^{n} \frac{1}{(i^2+i)^{3/4}} > 2-\frac{2}{\sqrt{n+1}}$$

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4 Two players, A (first player) and B, take alternate turns in playing a game using 2016 chips as follows: [i]the player whose turn it is, must remove s chips from the remaining pile of chips, where $s \in \{2, 4, 5\}$[/i]. No one can skip a turn. The player who at some point is unable to make a move (cannot remove chips from the pile) loses the game. Who among the two players can force a win on this game?

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5 Pentagon $ABCDE$ is inscribed in a circle. Its diagonals AC and BD intersect at F. The bisectors of $\angle BAC$ and $\angle CDB$ intersect at G. Let AG intersect BD at H, let DG intersect AC at I, and let EG intersect AD at J. If $FHGI$ is cyclic and

$$JA \cdot FC \cdot GH = JD \cdot FB \cdot GI,$$

prove that G, F and E are collinear.
