

IMO Shortlist 2009

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by April, orl, nsato, ZetaX

– Algebra

- 1** Find the largest possible integer k , such that the following statement is true:
Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are coloured, such that one is blue, one is red and one is white. Now, for every colour separately, let us sort the lengths of the sides. We obtain

$$\begin{aligned} b_1 \leq b_2 \leq \dots \leq b_{2009} & \quad \text{the lengths of the blue sides} \\ r_1 \leq r_2 \leq \dots \leq r_{2009} & \quad \text{the lengths of the red sides} \\ \text{and } w_1 \leq w_2 \leq \dots \leq w_{2009} & \quad \text{the lengths of the white sides} \end{aligned}$$

Then there exist k indices j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j .

Proposed by Michal Rolinek, Czech Republic

- 2** Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + b + c$. Prove that:

$$\frac{1}{(2a + b + c)^2} + \frac{1}{(a + 2b + c)^2} + \frac{1}{(a + b + 2c)^2} \leq \frac{3}{16}.$$

Proposed by Juhan Aru, Estonia

- 3** Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b , there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

Proposed by Bruno Le Floch, France

- 4** Let a, b, c be positive real numbers such that $ab + bc + ca \leq 3abc$. Prove that

$$\sqrt{\frac{a^2 + b^2}{a + b}} + \sqrt{\frac{b^2 + c^2}{b + c}} + \sqrt{\frac{c^2 + a^2}{c + a}} + 3 \leq \sqrt{2} \left(\sqrt{a + b} + \sqrt{b + c} + \sqrt{c + a} \right)$$

Proposed by Dzianis Pirshtuk, Belarus

- 5 Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f(x - f(y)) > yf(x) + x$$

Proposed by Igor Voronovich, Belarus

- 6 Suppose that s_1, s_2, s_3, \dots is a strictly increasing sequence of positive integers such that the sub-sequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \dots \quad \text{and} \quad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$$

are both arithmetic progressions. Prove that the sequence s_1, s_2, s_3, \dots is itself an arithmetic progression.

Proposed by Gabriel Carroll, USA

- 7 Find all functions f from the set of real numbers into the set of real numbers which satisfy for all x, y the identity

$$f(xf(x+y)) = f(yf(x)) + x^2$$

Proposed by Japan

– Combinatorics

- 1 Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

Proposed by Michael Albert, Richard Guy, New Zealand

- 2 For any integer $n \geq 2$, let $N(n)$ be the maximum number of triples (a_i, b_i, c_i) , $i = 1, \dots, N(n)$, consisting of nonnegative integers a_i, b_i and c_i such that the following two conditions are satisfied:

- $a_i + b_i + c_i = n$ for all $i = 1, \dots, N(n)$,

- If $i \neq j$ then $a_i \neq a_j$, $b_i \neq b_j$ and $c_i \neq c_j$

Determine $N(n)$ for all $n \geq 2$.

Proposed by Dan Schwarz, Romania

- 3** Let n be a positive integer. Given a sequence $\varepsilon_1, \dots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \dots, n-1$, the sequences a_0, \dots, a_n and b_0, \dots, b_n are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \quad \text{for each } i = 1, \dots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \quad \text{for each } i = 1, \dots, n-1.$$

Prove that $a_n = b_n$.

Proposed by Ilya Bogdanov, Russia

- 4** For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

Proposed by Gerhard Woeginger, Netherlands

- 5** Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighbouring buckets, empties them to the river and puts them back. Then the next round begins. The Stepmother goal's is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

Proposed by Gerhard Woeginger, Netherlands

- 6** On a 999×999 board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over. How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

Proposed by Nikolay Beluhov, Bulgaria

- 7** Let a_1, a_2, \dots, a_n be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \dots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths a_1, a_2, \dots, a_n in some order. Prove

that the order can be chosen in such a way that the grasshopper never lands on any point in M .

Proposed by Dmitry Khramtsov, Russia

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- 8** For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let r be the rightmost digit of n .
-If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of n by removing this rightmost digit 0.
-If $1 \leq r \leq 9$ we split the decimal representation of n into a maximal right part R that solely consists of digits not less than r and into a left part L that either is empty or ends with a digit strictly smaller than r . Then the decimal representation of $h(n)$ consists of the decimal representation of L , followed by two copies of the decimal representation of $R - 1$. For instance, for the number 17, 151, 345, 543, we will have $L = 17, 151, R = 345, 543$ and $h(n) = 17, 151, 345, 542, 345, 542$.
Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of h produces the integer 1 after finitely many steps.

Proposed by Gerhard Woeginger, Austria

– Geometry

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- 1** Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E , respectively. Let K be the incentre of triangle ADC . Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

Jan Vonk, Belgium, Peter Vandendriessche, Belgium and Hojoo Lee, Korea

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- 2** Let ABC be a triangle with circumcentre O . The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ , respectively, and let Γ be the circle passing through K, L and M . Suppose that the line PQ is tangent to the circle Γ . Prove that $OP = OQ$.

Proposed by Sergei Berlov, Russia

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- 3** Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y , respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram.
Prove that $GR = GS$.

Proposed by Hossein Karke Abadi, Iran

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- 4** Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F . The midpoints of AB and CD are G and H , respectively. Show that EF is tangent at E to the circle through the points E, G and H .

Proposed by David Monk, United Kingdom

- 5 Let P be a polygon that is convex and symmetric to some point O . Prove that for some parallelogram R satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets R and P , respectively.

Proposed by Witold Szczechla, Poland

- 6 Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P . Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of triangles ABP and CDP , respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2 , respectively. Prove that the perpendicular from E_1 on CD , the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.

Proposed by Eugene Bilopitov, Ukraine

- 7 Let ABC be a triangle with incenter I and let X, Y and Z be the incenters of the triangles BIC , CIA and AIB , respectively. Let the triangle XYZ be equilateral. Prove that ABC is equilateral too.

Proposed by Mirsaleh Bahavarnia, Iran

- 8 Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N . Denote by I_1, I_2 and I_3 the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on g .

Proposed by Nikolay Beluhov, Bulgaria

– Number Theory

- 1 Let n be a positive integer and let $a_1, a_2, a_3, \dots, a_k$ ($k \geq 2$) be distinct integers in the set $1, 2, \dots, n$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \dots, k - 1$. Prove that n does not divide $a_k(a_1 - 1)$.

Proposed by Ross Atkins, Australia

- 2 A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b , consider the polynomial P defined by $P(x) = (x + a)(x + b)$.
- (a) Prove that there exist distinct positive integers a and b such that all the number $P(1), P(2), \dots, P(50)$ are balanced.
- (b) Prove that if $P(n)$ is balanced for all positive integers n , then $a = b$.

Proposed by Jorge Típe, Peru

- 3** Let f be a non-constant function from the set of positive integers into the set of positive integers, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers a, b . Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c .

Proposed by Juhan Aru, Estonia

- 4** Find all positive integers n such that there exists a sequence of positive integers a_1, a_2, \dots, a_n satisfying:

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \leq k \leq n - 1$.

Proposed by North Korea

- 5** Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n -fold application of T .

Proposed by Jozsef Pelikan, Hungary

- 6** Let k be a positive integer. Show that if there exists a sequence a_0, a_1, \dots of integers satisfying the condition

$$a_n = \frac{a_{n-1} + n^k}{n} \text{ for all } n \geq 1,$$

then $k - 2$ is divisible by 3.

Proposed by Okan Tekman, Turkey

- 7** Let a and b be distinct integers greater than 1. Prove that there exists a positive integer n such that $(a^n - 1)(b^n - 1)$ is not a perfect square.

Proposed by Mongolia