IMO Shortlist 2011

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by Amir Hossein, orl, WakeUp

– Algebra

1 Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

Proposed by Fernando Campos, Mexico

2 Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers, such that for every positive integer n there exists an integer a with

$$\sum_{j=1}^{2011} j x_j^n = a^{n+1} + 1$$

Proposed by Warut Suksompong, Thailand

3 Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

Proposed by Japan

4 Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f)}_{k}(n)\ldots))$.

Proposed by Bojan Bai, Serbia

5 Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n+1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

Proposed by Canada

6 Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Proposed by Igor Voronovich, Belarus

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7 Let a, b and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}.$$

Proposed by Titu Andreescu, Saudi Arabia

– Combinatorics

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1 Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed.
Determine the number of ways in which this can be done.

Proposed by Morteza Saghafian, Iran

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2 Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

Proposed by Gerhard Wginger, Austria

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3 Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.
Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Proposed by Geoffrey Smith, United Kingdom

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4 Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

Proposed by Igor Voronovich, Belarus

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5 Let m be a positive integer, and consider a $m \times m$ checkerboard consisting of unit squares. At the centre of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in the opposite directions meet, they both turn 90° clockwise and continue moving with speed 1. When more than 2 ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard, or prove that such a moment does not necessarily exist.

Proposed by Toomas Krips, Estonia

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6 Let n be a positive integer, and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word, consisting of just letters a and/or b. Suppose that the minimal period N of W is greater than 2^n.

A finite nonempty word U is said to *appear* in W if there exist indices $k \le \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A finite word U is called *ubiquitous* if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

Proposed by Grigory Chelnokov, Russia

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7 On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?

Proposed by Ilya Bogdanov and Rustem Zhenodarov, Russia

– Geometry

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1 Let ABC be an acute triangle. Let ω be a circle whose centre L lies on the side BC. Suppose that ω is tangent to AB at B' and AC at C'. Suppose also that the circumcentre O of triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

Proposed by Hrmel Nestra, Estonia

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2 Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

Proposed by Alexey Gladkich, Israel

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3 Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA and AB. Prove that the midpoint of the segment EF lies on the line through the two intersections of ω_E and ω_F.

Proposed by Carlos Yuzo Shine, Brazil

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4 Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Proposed by Ismail Isaev and Mikhail Isaev, Russia

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5 Let ABC be a triangle with incentre I and circumcircle ω. Let D and E be the second intersection points of ω with AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.

Proposed by Irena Majcen and Kris Stopar, Slovenia

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6 Let ABC be a triangle with $AB = AC$ and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B and C at the point E inside the triangle ABC. The line BD intersects the circle through A, E and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incentre of triangle KAB.

Proposed by Jan Vonk, Belgium and Hojoo Lee, South Korea

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7 Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with centre O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

Proposed by Japan

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8 Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Proposed by Japan

– Number Theory

1 For any integer $d > 0$, let $f(d)$ be the smallest possible integer that has exactly d positive divisors (so for example we have $f(1) = 1, f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f\left(2^k\right)$ divides $f\left(2^{k+1}\right)$.

Proposed by Suhaimi Ramly, Malaysia

2 Consider a polynomial $P(x) = \prod_{j=1}^{9}(x+d_j)$, where $d_1, d_2, \ldots d_9$ are nine distinct integers. Prove that there exists an integer N, such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

Proposed by Luxembourg

3 Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself, such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

Proposed by Mihai Baluna, Romania

4 For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n, such that all the differences

$$t(n+a) - t(n); t(n+a+1) - t(n+1), \ldots, t(n+2a-1) - t(n+a-1)$$

are divisible by 4.

Proposed by Gerhard Wginger, Austria

5 Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

Proposed by Mahyar Sefidgaran, Iran

6 Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients, such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)} - 1$ divides $3^{P(n)} - 1$. Prove that $Q(x)$ is a constant polynomial.

Proposed by Oleksiy Klurman, Ukraine

7 Let p be an odd prime number. For every integer a, define the number $S_a = \sum_{j=1}^{p-1} \frac{a^j}{j}$. Let $m, n \in \mathbb{Z}$, such that $S_3 + S_4 - 3S_2 = \frac{m}{n}$. Prove that p divides m.

Proposed by Romeo Metrovi, Montenegro

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8 Let $k \in \mathbb{Z}^+$ and set $n = 2^k + 1$. Prove that n is a prime number if and only if the following holds: there is a permutation $a_1, \dots, a_{n-1}$ of the numbers $1, 2, \dots, n-1$ and a sequence of integers $g_1, \dots, g_{n-1}$, such that n divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \dots, n-1\}$, where we set $a_n = a_1$.

Proposed by Vasily Astakhov, Russia
