Albania Team Selection Test 2012

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by Olemissmath

1 Find the greatest value of the expression

$$\frac{1}{x^2-4x+9}+\frac{1}{y^2-4y+9}+\frac{1}{z^2-4z+9}$$

where x, y, z are nonnegative real numbers such that $x+y+z=1$.

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2 It is given an acute triangle ABC , $AB \neq AC$ where the feet of altitude from A its H. In the extensions of the sides AB and AC (in the direction of B and C) we take the points P and Q respectively such that $HP=HQ$ and the points B, C, P, Q are concyclic.
Find the ratio $\frac{HP}{HA}$.

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3 It is given the equation $x^4-2ax^3+a(a+1)x^2-2ax+a^2=0$.
a) Find the greatest value of a, such that this equation has at least one real root.
b) Find all the values of a, such that the equation has at least one real root.

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4 Find all couples of natural numbers (a,b) not relatively prime ($\gcd(a,b) \neq 1$) such that

$$\gcd(a,b)+9\,\mathrm{lcm}[a,b]+9(a+b)=7ab.$$

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5 Let $f:\mathbb{R}^+ \to \mathbb{R}^+$ be a function such that:

$$x,y>0 \qquad f(x+f(y))=yf(xy+1).$$

a) Show that $(y-1)*(f(y)-1) \leq 0$ for $y>0$.
b) Find all such functions that require the given condition.
