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Day 1 April 27th

1 A *permutation* of the set of positive integers $[n] = \{1, 2, ..., n\}$ is a sequence $(a_1, a_2, \ldots, a_n)$ such that each element of $[n]$ appears precisely one time as a term of the sequence. For example, $(3, 5, 1, 2, 4)$ is a permutation of $[5]$. Let $P(n)$ be the number of permutations of $[n]$ for which ka_k is a perfect square for all $1 \leq k \leq n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

2 Let $n > 1$ be an integer. Find, with proof, all sequences $x_1, x_2, \ldots, x_{n-1}$ of positive integers with the following three properties:
(a). $x_1 < x_2 < \cdots < x_{n-1}$;
(b). $x_i + x_{n-i} = 2n$ for all $i = 1, 2, \ldots, n-1$;
(c). given any two indices i and j (not necessarily distinct) for which $x_i + x_j < 2n$, there is an index k such that $x_i + x_j = x_k$.

3 Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Day 2 April 28th

4 A triangle is called a parabolic triangle if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

5 Two permutations $a_1, a_2, \ldots, a_{2010}$ and $b_1, b_2, \ldots, b_{2010}$ of the numbers $1, 2, \ldots, 2010$ are said to *intersect* if $a_k = b_k$ for some value of k in the range $1 \leq k \leq 2010$. Show that there exist 1006 permutations of the numbers $1, 2, \ldots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

6 Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.