IberoAmerican Olympiad For University Students 1998

www.artofproblemsolving.com/community/c3979

by Jorge Miranda

1 The definite integrals between 0 and 1 of the squares of the continuous real functions $f(x)$ and $g(x)$ are both equal to 1.
Prove that there is a real number c such that

$$f(c) + g(c) \leq 2$$

---

2 In a plane there is an ellipse E with semiaxis a, b. Consider all the triangles inscribed in E such that at least one of its sides is parallel to one of the axis of E.
Find both the locus of the centroid of all such triangles and its area.

---

3 The positive divisors of a positive integer n are written in increasing order starting with 1.

$$1 = d_1 < d_2 < d_3 < \cdots < n$$

Find n if it is known that:
i. $n = d_{13} + d_{14} + d_{15}$
ii. $(d_5 + 1)^3 = d_{15} + 1$

---

4 Four circles of radius 1 with centers A, B, C, D are in the plane in such a way that each circle is tangent to two others. A fifth circle passes through the center of two of the circles and is tangent to the other two.
Find the possible values of the area of the quadrilateral $ABCD$.

---

5 A sequence of polynomials $\{f_n\}_{n=0}^{\infty}$ is defined recursively by $f_0(x) = 1$, $f_1(x) = 1 + x$, and

$$(k+1)f_{k+1}(x) - (x+1)f_k(x) + (x-k)f_{k-1}(x) = 0, \quad k = 1, 2, \ldots$$

Prove that $f_k(k) = 2^k$ for all $k \geq 0$.

---

6 Take the following differential equation:

$$3(3 + x^2)\frac{dx}{dt} = 2(1 + x^2)^2 e^{-t^2}$$

If $x(0) \leq 1$, prove that there exists $M > 0$ such that $|x(t)| < M$ for all $t \geq 0$.

---

7 Some time ago there was a war across the world. In the plane n lines are moving, with the regions contained by the lines being the territories of the countries at war. Each line moves parallel to itself with constant speed (each with its own speed), and no line can reverse its direction. Some of the original countries disappeared (a country disappears iff its area is converted to zero) and within the course of the time, other countries appeared.

After some time, the presidents of the existing countries made a treaty to end the war, created the United Nations, and all borders ceased movement. The UN then counted the total numbers of sovereign states that were destroyed and the existing ones, obtaining a total of k.

Prove that $k \leq \frac{n^3+5n}{6} + 1$. Is is possible to have equality?