IberoAmerican Olympiad For University Students 2005

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1 Let $P(x,y)=(x^2y^3,x^3y^5)$, $P^1=P$ and $P^{n+1}=P\circ P^n$. Also, let $p_n(x)$ be the first coordinate of $P^n(x,x)$, and $f(n)$ be the degree of $p_n(x)$. Find

$$\lim_{n\to\infty} f(n)^{1/n}$$

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2 Let A,B,C be real square matrices of order n such that $A^3=-I$, $BA^2+BA=C^6+C+I$ and C is symmetric. Is it possible that $n=2005$?

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3 Consider the sequence defined recursively by $(x_1,y_1)=(0,0)$,

$(x_{n+1},y_{n+1})=\left(\left(1-\frac{2}{n}\right)x_n-\frac{1}{n}y_n+\frac{4}{n},\left(1-\frac{1}{n}\right)y_n-\frac{1}{n}x_n+\frac{3}{n}\right)$.

Find $\lim_{n\to\infty}(x_n,y_n)$.

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4 A variable tangent t to the circle C_1, of radius r_1, intersects the circle C_2, of radius r_2 in A and B. The tangents to C_2 through A and B intersect in P.
Find, as a function of r_1 and r_2, the distance between the centers of C_1 and C_2 such that the locus of P when t varies is contained in an equilateral hyperbola.

Note: A hyperbola is said to be *equilateral* if its asymptotes are perpendicular.

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5 Arnaldo and Bernaldo play a game where they alternate saying natural numbers, and the winner is the one who says 0. In each turn except the first the possible moves are determined from the previous number n in the following way: write

$$n=\sum_{m\in O_n}2^m;$$

the valid numbers are the elements m of O_n. That way, for example, after Arnaldo says $42=2^5+2^3+2^1$, Bernaldo must respond with 5, 3 or 1.

We define the sets $A,B\subset\mathbb{N}$ in the following way. We have $n\in A$ iff Arnaldo, saying n in his first turn, has a winning strategy; analogously, we have $n\in B$ iff Bernaldo has a winning strategy if Arnaldo says n during his first turn. This way,

$$A=\{0,2,8,10,\cdots\},B=\{1,3,4,5,6,7,9,\cdots\}$$

Define $f:\mathbb{N}\to\mathbb{N}$ by $f(n)=|A\cap\{0,1,\cdots,n-1\}|$. For example, $f(8)=2$ and $f(11)=4$.
Find

$$\lim_{n\to\infty}\frac{f(n)\log(n)^{2005}}{n}$$

6 A smooth function $f : I \to \mathbb{R}$ is said to be *totally convex* if $(-1)^k f^{(k)}(t) > 0$ for all $t \in I$ and every integer $k > 0$ (here I is an open interval).

Prove that every totally convex function $f : (0, +\infty) \to \mathbb{R}$ is real analytic.

Note: A function $f : I \to \mathbb{R}$ is said to be *smooth* if for every positive integer k the derivative of order k of f is well defined and continuous over $\mathbb{R}$. A smooth function $f : I \to \mathbb{R}$ is said to be *real analytic* if for every $t \in I$ there exists $\epsilon > 0$ such that for all real numbers h with $|h| < \epsilon$ the Taylor series

$$\sum_{k \geq 0} \frac{f^{(k)}(t)}{k!} h^k$$

converges and is equal to $f(t + h)$.

7 Prove that for any integers n, p, $0 < n \leq p$, all the roots of the polynomial below are real:

$$P_{n,p}(x) = \sum_{j=0}^{n} \binom{p}{j} \binom{p}{n-j} x^j$$